Preface

Education is the key to development. A progressively improved education system largely determines the pace and the quality of national development. To reflect the hopes and aspirations of the people and the socio-economic and cultural reality in the context of the post independent Bangladesh, new textbooks were introduced on the beginning of the 1980s following the recommendations of the National Curriculum and Textbook Committee.

In 1994, in accordance with the need for change and development, the textbooks of Lower Secondary, Secondary and Higher Secondary were revised and modified. The textbooks from classes VI to IX were written in 1995. In 2000, almost all the textbooks were rationally evaluated and necessary revision were made. In 2008, the Ministry of Education formed a Task Force for Education. According to the advice and guidance of the Task Force, the cover, spelling and information in the textbooks were updated and corrected.

To make assessment more meaningful and in accordance with the need of the curriculum, Creative Questions and Multiple Choice Questions are given at the end of each chapter. It is hoped that this will reduce the dependency of students on rote memorization. The students will be able to apply the knowledge they have gained to judge, analyses and evaluate real life situation.

The textbook of Lower Secondary, Secondary and Higher Secondary have been revised to modernize the study of Mathematics. The study of Arithmetic has been limited to class Eight and Algebra has been introduced. It is hoped that students will be able to solve mathematical problems with the use of Algebra. For this reason Algebra has been introduced from class Six. Mathematics is a subject that has to be practical. It cannot be memorized. For more practice creative questions and multiple choice questions have been included in the exercises together with traditional questions.

This book of Algebra for class IX & X is the English Version of the original textbook entitled 'Maydhamic Bijganit' written in Bangla.

We know that, curriculum development is a continuous process on which textbooks are written. Any logical and formative suggestions for improvement will be considered with care. On the event of the golden jubilee of the Independence of Bangladesh in 2021, we want to be a part of the ceaseless effort to build a prosperous Bangladesh.

In spite of sincere efforts in translation, editing and printing some inadvertent errors and omissions may be found in the book. However, our efforts to make it more refined and impeccable will continue.

I thank those who have assisted us with their intellect and effort in the writing, editing and rational evaluation of this book. We hope that the book will be useful for the students for whom it is written.

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Chairman
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Dhaka
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Chapter 1
Set

The use of set as a tool in modern mathematics is extensive. German Mathematician George Cantor (1844–1918) first introduced the notion of sets. His investigations of infinite sets led to the creation of a new branch of Mathematics which is at present known as Set Theory.

Set : In daily life, the word set is often used to denote a group or cluster or collection of objects. In mathematical discussions too, the word set is used to represent a well defined collection of objects of real world or our intuition. Taking set as an undefined notion like many basic concepts of geometry, we shall use the word set only to mean a well defined collection' of objects. By well defined, it is understood that the elements which are included or not included in the set and this is clearly determined.

Capital letters of English alphabets such as A, B, C, D, X, Y etc. are usually used to represent sets and small letters a, b, c, x, y etc. are used to represent the elements of a set. For example, let A be the set of all even numbers. Then 6 is a member of A. It is written as $6 \in A$ and read as 6 is in A or 6 belongs to A. 5 is not a member of A. It is written as $5 \notin A$ and read as 5 is not in A or 5 does not belong to A. A member of a set is also called an element of the set. There are two methods to express sets.

1. Tabular Method : In this method, all the elements of a set are written within curly brackets { } and the commas are being used to separate the elements. For example,
   
   A = \{2, 3, 5, 7, 11, 13, 17\}
   B = \{b, o, y\}
   C = \{1, 3, 5, 7, 9, ............\}

   Dots (....) are used to represent those elements which are not mentioned in the set. Tabular Method is also called Roster Method.

2. Set Builder Method : In this method, a set is described by mentioning the common characteristics of the elements of the set such as \( A = \{ x : x \text{ is an even natural number} \} \). Here ':' is used to mean 'such that'. In the above example, A is the set of all x such that x is an even natural number. As the rule for determining the elements is given in this method, that is why this method is also called the Rule Method.
Example 1. S denotes the set of all divisions of Bangladesh. Express S in roster method and set builder method.

Solution: In roster method, S = {Dhaka, Chittagong, Khulna, Rajshahi, Barisal, Sylhet}. In set builder method, S = \{ x : x is a division of Bangladesh \}.

Finite Set: A set with countable number of elements is called a finite set. For example, B = \{ a, b, c \} is a finite set.

Infinite Set: A set with uncountable number of elements is called an infinite set. The set of all natural numbers N = \{ 1, 2, 3, 4, ........... \} is an infinite set.

Equal Set: Two sets A and B are said to be equal if they have equal number of elements and it is written A = B. For example, \{ 2, b, c \} = \{ b, c, 2 \}. It is to be noted that, a set remains unchanged if the orders of elements are changed or if any elements are repeated, such as, \{ 1, 2, 2, 3, 1 \} = \{ 1, 2, 3 \}.

Subset: If every element of the set A belongs to set B, then A is said to be a subset of B. It is written as A \subset B and read as A is a subset of B. For example, if A = \{ 2, 4, 6, 8 \} and B = \{ 1, 2, 3, 4, 5, 6, 7, 8, 9 \}, then A \subset B. A is a subset of A.

Proper Subset: If every element of A belongs to B and if there is an element of B which does not belong to A, then A is called a proper subset of B. This is denoted by A \nsubseteq B. A itself is not a proper subset of A.

If a set A is given and a set B is constructed with some elements of A, then B is a subset of A. In constructing a subset of A, the elements are chosen from A satisfying one or more conditions. For example, five subsets of the set of natural numbers N are constructed. Here, it is understood from \( \frac{a}{b} \) that the natural number a to completely divisible by natural number b.

<table>
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<th>In notation</th>
<th>In words</th>
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<tr>
<td>A = { x \in N: x&lt;10 }</td>
<td>The set of natural numbers less than 10</td>
</tr>
<tr>
<td>B = { x \in N: x/16 }</td>
<td>The set of natural numbers which are factors of 16</td>
</tr>
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<td>C = { x \in N: 7/x }</td>
<td>The set of natural numbers which are multiples of 7</td>
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<td>D = { x \in N: x &lt; 30 and x is a prime number }</td>
<td>The set of those prime numbers which are less than 30.</td>
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<tr>
<td>E = { x \in N : x^2 &gt;10 and x^3 &lt; 100 }</td>
<td>The set of natural numbers whose squares are greater than 10 and cubes are less than 100.</td>
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The numbers which are the elements of the above mentioned subsets of N can easily be determined from the given conditions.
A = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}; \quad B = \{1, 2, 4, 8, 16\};
C = \{7, 14, 21, 28, \ldots \}; \quad D = \{2, 3, 5, 7, 11, 13, 17, 19, 23, 29\};
E = \{4\}

Of these sets, C is an infinite set that is, C has innumerable elements and E is a singleton set, that is E has just one element. We observe that E \subset A, E \subset B but E \not\subset C, E \not\subset D.

**Empty Set:** The set \{ x \in \mathbb{N}: x < 9 \text{ and } x > 10\} dose not have any element. Because there is no natural number which is less than 9 and greater than 10. Such set is called Empty Set and it is written by the symbol \{ \} or \emptyset.

Other examples of empty sets can be cited. For example, \{x \in \mathbb{N} : 23 < x < 29 \text{ and } x \text{ is prime number}\}.

**Universal Set:** In mathematics, all sets under discussion are subsets of a fixed set. In this context, the fixed set is called the Universal Set of all sets under discussions. Generally, the symbol U is used for the universal set. But any other symbol may be used.

**Union of Sets:** The set consisting of all the elements of two sets is called the union of those two sets. The union of sets A and B is denoted by A \cup B and it is read as \( A \) union \( B \). In set-builder notation, the definition of A \cup B is:

\[ A \cup B = \{ x : x \in A \text{ or } x \in B \}. \]

**Example 2.** Let A = \{ 1, 2, 3, 4\} and B = \{ 2, 4, 6, 8 \} be two sets.

Then \( A \cup B = \{1, 2, 3, 4, 6, 8\} \).

Here 2 and 4 are in both the sets. In the union, 2 and 4 are listed once without the repetition.

**Intersection of Sets:** The set consisting of the common elements of two sets is called intersection of those two sets. The intersection of sets A and B is denoted by A \cap B and is read as \( A \) intersection \( B \). The definition of A \cap B in set builder notation is A \cap B = \{ x : x \in A \text{ and } x \in B \}.

**Example 3.** If A = \{ 0, 1, 2, 3 \}, B = \{ 0, 3, 4, 5 \}, then determine A \cup B and A \cap B.

**Solution:**

\[ A \cup B = \{ 0, 1, 2, 3 \} \cup \{ 0, 3, 4, 5 \} = \{0, 1, 2, 3, 4, 5\} \]
\[ A \cap B = \{ 0, 1, 2, 3 \} \cap \{ 0, 3, 4, 5 \} = \{3\}. \]

**Example 4.** If C = \{ 1, 2, 3, 4 \}, D = \{ 0, 5, 6, 8 \}, then determine C \cup D and C \cap D.
Solution: \[ C \cup D = \{1, 2, 3, 4\} \cup \{0, 5, 6, 8\} = \{1, 2, 3, 4, 0, 5, 6, 8\} \]
\[ C \cap D = \{1, 2, 3, 4\} \cap \{0, 5, 6, 8\} = \emptyset. \]

Disjoint Set: If two sets do not have any common element, then the sets are said to be disjoint. If the two sets \( A \) and \( B \) are disjoint, then \( A \cap B = \emptyset \).

Venn Diagram (Jhon Venn: 1834 ð 1883): When operations on sets such as union, intersection etc. and the rules related to them are represented through geometric figures, such figures are called Venn Diagrams. In Venn diagrams, the sets under consideration are shown as plane geometric figures. Usually, the universal set is represented by a rectangle. Circles or triangles are used to represent subsets.

Complementary Sets: Let \( A \) and \( B \) be two sets. The set of those elements of \( A \) which do not belong to \( B \) is said to be the complement of \( B \) relative to \( A \) and it is denoted by \( A \setminus B \). \( A \setminus B \) is read as \( A \) minus \( B \).

\[ A \setminus B = \{x \in A : x \notin B\}. \]

To denote \( A \setminus B \), the notation \( A \setminus B \) is also used.
Complement of \( A \) relative to \( B \) is \( B \setminus A = \{x \in B : x \notin A\} \).

If in some context \( U \) is the universal set, then \( U \setminus A \) is denoted in short by \( A^c \) and \( A^c \) is simply called the complement of \( A \).

\[ A^c = \{x \in U : x \notin A\}. \]

In words: \( A^c \) consists of those elements of the universal set which are not the elements of \( A \).

\( A^c \) is shown in Venn diagram. Here, the universal set and its subset \( A \) are shown by a rectangle and a circle respectively. Complement \( A^c \) of \( A \) is shaded.

Example 5. The sets \( A \) and \( B \) are the sets of the factors of 108 and 87 respectively. Determine \( A \) and \( B \).

Solution: The factors of 108 are 1, 2, 3, 4, 6, 9, 12, 18, 27, 36, 54, 108.
Therefore, \( A = \{1, 2, 3, 4, 6, 9, 12, 18, 27, 36, 54, 108\} \).
The factors of 87 are 1, 3, 29, 87.
Therefore, \( B = \{1, 3, 29, 87\} \).
**Example 6.** Determine the set of natural numbers which divides 346 and 556 with 31 as remainder in each case.

**Solution:** The numbers which divide 346 and 556 with 31 as remainder are greater than 31. Such numbers are the common factors of \((346 ÷ 31) = 315\) and \((556 ÷ 31) = 525\).

Let \(A\) be the set of factors of 315 greater than 31 and \(B\) be the set of factors of 525 greater than 31. Then

\[
A = \{35, 45, 63, 105, 315\} \\
B = \{35, 75, 105, 525\}
\]

\[\therefore\text{the required set is } A \cap B = \{35, 105\}.
\]

**Example 7.** In an examination, 80% of examinees passed in Mathematics and 70% of examinees passed in Bangla. 60% passed in both the subjects. Find the percentage of the students who failed in both the subjects.

**Solution:** Consider the adjoining Venn diagram. Here the rectangular region denotes the set \(E\) of 100 examinees. The circular regions indicated by \(M\) and \(B\) denote the set of examinees passed in Mathematics and Bangla respectively. In the diagram we have four disjoint sets marked \(P_1, P_2, P_3, P_4\). Here,

\[
P_2 = M \cap B \text{ is set of examinees passed in both subjects and number of elements } = 60
\]

\[
P_1 = M \setminus P_2 \text{ is set of examinees passed in Mathematics and number of elements } = 80 \setminus 60 = 20
\]

\[
P_3 = B \setminus P_2 \text{ is set of examinees passed in Bangla only and number of elements } = 70 \setminus 60 = 10
\]

\[
M \cup B = P_1 \cup P_2 \cup P_3 \text{ is set of examinees passed in one or both subjects and number of elements } = 20 + 60 + 10 = 90
\]

\[\therefore\text{ } P_4 = E \setminus (M \cup B) \text{ is set of examinees failed in both subjects and number of elements } = 100 \setminus 90 = 10.
\]

\[\therefore\text{10% of examinees failed in both subjects.}\]
Exercise D 1.1

1. If \( A = \{1, 2, 3, 4, 5, 6\} \), then construct true sentences filling the blank spaces using the symbols \( \in \) or \( \notin \):
   (i) 5 \( \in \) A (ii) 8 \( \notin \) A (iii) 4 \( \in \) A (iv) 0 \( \notin \) A (v) 10 \( \notin \) A.

2. Construct true sentences putting \( \subset \) or \( \not\subset \) in the blank spaces:
   (i) \{2, 3\} \( \subset \) \{1, 2, 3, 4\} (ii) \{a, b, c\} \( \not\subset \) \{b, c, d\}
   (iii) \{x : x is a student of class IX of your school\} \( \not\subset \) \{x : x is a student of your school\}
   (iv) \{x : x is an even natural number\} \( \subset \) \{x : x is an integer\}.

3. Determine the following sets in Roster method:
   (i) \{x \in \mathbb{N} : x^2 > 15 and x^3 < 100\}
   (ii) \{x \in \mathbb{N} : x is an integer and x^2 < 13\}
   (iii) \{x \in \mathbb{N} : 6 < x < 7\}
   (iv) \{x \in \mathbb{N} : x < 10 and x is even\}
   (v) \{x \in \mathbb{N} : x is a factor of 42\}
   (vi) \{x \in \mathbb{N} : x < 19 and x is a multiple of 3\}

4. (i) A and B are the sets of factors of 315 and 525 respectively. Determine A and B.
   (ii) Determine the set of natural numbers which divides 311 and 419 with 23 as remainder in each cases.
   (iii) Determine the set of natural numbers which divides 105 and 147 with 35 as remainder in each case.

5. If \( A = \{1, 2, 3\} \) and \( B = \{3, a, b\} \), determine \( A \cup B \) and \( A \cap B \).

6. Write down three proper subsets of \( \{\emptyset, 0, 1, 2\} \), each of which consists of three elements.

7. If \( X = \{1, 2, 3\} \) and \( Y = \{4, 5, 6\} \), determine \( X \cap Y \).

8. If \( A = \{1, 2, 3\} \) and \( B = \emptyset \), determine \( A \cup B \) and \( A \cap B \).

9. If \( U = \{1, 2, 3, 4, 5, 6\} \), \( A = \{1, 3, 5\} \), \( B = \{2, 4, 6\} \) and \( C = \{2, 3, 4, 5\} \), then determine the following sets:
   (i) \( A \triangle B \) (ii) \( C \triangle B \) (iii) \( A\triangle \) (iv) \( B\triangle \) (v) \( A\triangle \cup C\triangle \) (vi) \( A\triangle \cap B \)
10. Verify the following statements in the case of sets in problem 9 above:
   (i) \((A \cup B)\cap = A\cap B\cap\)
   (ii) \((B \cap C)\cap = B\cap C\cap\)
   (iii) \((A \cup B) \cap C = (A \cap C) \cup (B \cap C)\)
   (iv) \((A \cap B) \cup C = (A \cup C) \cap (B \cup C)\)
   (v) \(A \cap B = (A \setminus B) \cup (B \setminus A) \cup (A \cap B)\).

11. If \(A= \{1, 2, 3\}, B = \{2, 4, 6\}, C = \{1, 4, 7\}\), show that
   \((A \cup B) \cup C = A \cup (B \cup C)\) and \((A \cap B) \cap C = A \cap (B \cap C)\).
   \[A \cup B \cup C \\text{and} \ A \cap B \cap C \text{are written to express union and intersection of there sets.}\]

12. There were 100 students in a class. In the annual examination, 94 students passed in Bangla, 80 students passed in Mathematics and 75 students passed in both the subjects. Express the information with the help of Venn diagram. How many of them failed in both the subjects?

13. In a class of 25 students, each student is allowed to take at least one subject either of Computer Science or of Higher Mathematics. It is found that 12 students have taken Computer Science of which 8 students did not take Higher Mathematics. Determine the number of students who take both the subjects and who take only Higher Mathematics.

**Power Set**

Let \(A\) be a set. The set of all subsets of \(A\) is the power set of \(A\) and it is denoted by \(P(A)\).

**Example 8.** (a) If \(A = \{a\}\), then determine \(P(A)\).
   (b) If \(A = \emptyset\), then determine \(P(\emptyset)\).

**Solution:** (a) The subsets of \(A\) are \(\{a\}, \emptyset\).
   \[\therefore P(A) = \{\{a\}, \emptyset\}\]
   (b) \(P(\emptyset) = \{\emptyset\}\). It is to be noted that, the set of empty set is not power set.

**Example 9.** If \(A = \{2, 3\}\), then determine \(P(A)\).

**Solution:** The subsets of \(A\) are \(\{2, 3\}, \{2\}, \{3\}, \emptyset\).
   \[\therefore P(A) = \{\{2, 3\}, \{2\}, \{3\}, \emptyset\}\]

**Example 10.** If \(A = \{2, a, e\}\), then write down all the elements of \(P(A)\).

**Solution:** Elements of \(P(A)\) are all the possible subsets of \(A\). These are \(\emptyset, \{2\}, \{a\}, \{e\}, \{2, a\}, \{2, e\}, \{a, e\}, \{2, a, e\}\).
Example 11. If \( A = \{a, b, c, d\} \), what is the number of elements of \( P(A) \)?

**Solution:** The elements of \( P(A) \) are
\[
\emptyset, \{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, \{a, b, c, d\}.
\]
So there are 16 elements in \( P(A) \).

**N.B.** If the number of elements of \( A \) is \( n \), then the number of the elements of \( P(A) \) is \( 2^n \).

**Ordered Pair:** Of the elements \( x, y \) of any set, if \( x \) is taken as first element and \( y \) is taken as the second element then we get an ordered pair \((x, y)\). \((x, y)\) is called an ordered pair instead of simple a pair. Because in it the terms are arranged as the first position and the second position. The ordered pairs \((x, y)\) and \((a, b)\) are equal, i.e., \((x, y) = (a, b)\) if and only if \(x = a\) and \(y = b\). If \(x\) and \(y\) are different elements then \((x, y) \neq (y, x)\). \((x, y) = (y, x)\) if and only if \(x = y\).

In graphs, \((x, y)\) specifies a point in the plane whose abscissa is \(x\) and ordinate is \(y\). The ordered pair \((3, 4)\) and \((4, 3)\) specifies two points in a graph. Hence the ordered pair \((3, 4)\) and \((4, 3)\) are different. But as sets \(\{3, 4\} = \{4, 3\}\), because, a set remains the same when the position of the elements are changed. An ordered pair and a set of two elements are never equal. It is to be noted that the ordered pair with first element \(a\) and second element \(b\) is expressed by writing \(a\) first and then \(b\) within first brackets, i.e. in the form \((a, b)\). \((a, a)\) is an ordered pair where both the first and the second elements are \(a\). It is to be noted that \(\{a, a\} = \{a\}\). But \((a, a,)\) cannot be written as \((a)\).

**Example 12.** If \((x + y, 0) = (1, x \neq y)\), then find the value of \(x\) and \(y\). Also find \((x, y)\).

**Solution:** According to the problem,
\[
x + y = 1 \quad \text{..................................................(i)}
\]
and \(0 = x \neq y\), \(\text{or, } x \neq y = 0 \quad \text{.................(ii)}
\]

Adding (i) and (ii), \(2x = 1\) or, \(x = \frac{1}{2}\)

Also from (ii), \(x = y\)

Hence, \(x = y = \frac{1}{2}\)

\[\therefore \ (x, y) = \left(\frac{1}{2}, \ \frac{1}{2}\right)\]
**Cartesian Product** : Suppose, the outside of a car is to be coloured by red, green or blue and inside is to be coloured white or yellow. If \( A \) is the set of colours for outside and \( B \) is the set of colours for inside then \( A = \{r, g, b\} \) and \( B = \{w, y\} \), where \( r, g, b, w, y \) stand for red, green, blue, white and yellow colour respectively. If the colour of outside is considered as first element and the colour of inside is considered as second element then the probable arrangements of the colours are six ordered pairs.

\((r, w), (r, y), (g, w), (g, y), (b, w), (b, y)\).

For the set of these ordered pairs we write,

\( A \times B = \{(r, w),(r, y), (g, w), (g, y), (b, w), (b, y)\} \).

This is an example of Cartesian product. In the above mentioned example, the car can be coloured by \(3 \times 2 = 6\) ways.

**Example 13.** Shuzan and Abid are coming to Dhaka together by launch. It is learnt from their discussion that Shuzan will stay in the house of Mama (uncle) and Khala (aunt) and Abid will stay in the house of Chacha (uncle), Fufu (aunt) and Dada (grandfather). Describe their possible stay in Dhaka by means of ordered pairs. In the ordered pairs, Shuzan's stay is to be considered first.

**Solution** : Let, the set of different places of Shuzan's stay = \( A \),

the set of different places of Abid's stay = \( B \).

Let us consider further that \( M, K, C, F \) and \( D \) stand for Mama, Khala, Chacha, Fufu and Dada's residence respectively. Then the set of their possible stay is

\( A \times B = \{(M, C), (M, F), (M, D), (K, C), (K, F), (K, D)\} \)

**Cartesian Product** : Let \( A \) and \( B \) be any set. The set of all ordered pairs of the elements of the set \( A \) and the set \( B \) is the Cartesian product set of \( A \) and \( B \). The Cartesian product of the set \( A \) and \( B \) is denoted by \( A \times B \) and it is read as \( A \) cross \( B \). In set - builder form, it can be described as

\( A \times B = \{(x, y) : x \in A, y \in B\} \).

For any set \( S \), \( S \times S = \{(x, y) : x, y \in S\} \)

**Example 14.** If \( S = \{1, 2, 4\} \), then determine \( S \times S \).

**Solution** : \( S \times S = \{(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (2, 4), (4, 1), (4, 2), (4, 4)\} \).

**Example 15.** If \( A = \{3, 4, 5\} \), \( B = \{4, 5, 6, 7\} \) and \( C = \{a, b\} \), then determine \((A \cap B) \times C\).

**Solution** : \((A \cap B) = \{3, 4, 5\} \cap \{4, 5, 6, 7\} = \{4, 5\}\)

\( \therefore (A \cap B) \times C = \{4, 5\} \times \{a, b\} = \{(4, a), (4, b), (5, a), (5, b)\} \).
Exercise 1.2

1. If \( B = \{1, 2\} \), then find \( P(B) \).
2. If \( C = \{a, b, c\} \), then find \( P(C) \).
3. If \((x + y, 1) = (3, x \Delta y)\), then find the value of \( x \) and \( y \).
4. If \((x \Delta 1, y + 2) = (y \Delta 2, 2x + 1)\), then find \((x, y)\).
5. If \( A = \{0, 1\} \) and \( B = \{1, 2\} \), then find \( A \times B \) and \( B \times A \).
6. If \( A = \{a, b, c\}, B = \{p, q\} \), then find \( A \times B \) and \( B \times A \).
7. If \( A = \{a, b\}, B = \{2, 3\} \) and \( C = \{3, 4\} \), then find \( A \times (B \cup C) \) and \( A \times (B \cap C) \).
8. If \( A = \{\emptyset, 1\} \) and \( B = \{0\} \), then find \( A \times B \) and \( B \times A \).
9. If \( A = \{0, 1\}, B = \left[\frac{1}{2}, \frac{1}{3}\right] \), then find \( A \times B \).

10. Abul and Babul are two friends. They have decided that on a particular day in tiffin period; Abul will go either to the canteen or to the library or to the playground and Babul will go either to the library or to the garden. Describe their possible stays on that day by means of ordered pairs. Abul's position is to be considered first.

[Hints: Denote the canteen, the library, the playground and garden by \(c, l, f\) and \(g\) respectively and the set of places of Abul's stay by \(A\) and set of places of Babul's stay by \(B\) ]

11. In a class Anu, Shuman and Mim are candidate for captain and Rahi and Masha are the candidate for vice-captain. Keeping the name of captain first, express the probable election alliance through Cartesian Product set.

12. The set of those players of national cricket team is \( A = \{Akram, Bulbul, Nannu\} \). Form probable pair of captain and vice-captain from them and express through Cartesian Product set.
**Multiple Choice Questions (MCQ):**

1. If \(A = \{0, 1, 2\}\) and \(B = \{\emptyset, 0, 1\}\), then which one of the followings is the correct value of \(A \cup B\)?
   
   A. \(\{0, 1\}\)  
   B. \(\{0, 1, 2\}\)  
   C. \(\emptyset, 0, 1\)  
   D. \(\emptyset, 0, 1, 2\)  

2. If set \(P\) is a proper subset of set \(Q\), then which one is the correct relations as shown below?
   
   A. \(P \subset \neq Q\)  
   B. \(P \subseteq Q\)  
   C. \(Q \subset P\)  
   D. \(P \subseteq Q\)  

3. Roll numbers of some students of Grade IX are defined as \(A\) which is factors of 12. Which of the following represents set \(A\)?
   
   A. \(\{12, 24, 36, 48, \ldots\}\)  
   B. \(\{1, 2, 3, 4, 6, 12\}\)  
   C. \(\{2, 3, 4, 6\}\)  
   D. \(\{2, 3, 4, 6, 12\}\)  

4. Look at the following mathematical sentences:
   
   i. \(A \cup B = \{x : x \in A \text{ or } x \in B\}\)  
   ii. \(A \times B = \{(x, y) : x \in A \text{ and } y \in B\}\)  
   iii. \(A' = \{x : x \in U \text{ and } x \in A\}\)
   
   Which of the above sentences are correct?
   
   A. i and ii  
   B. i and iii  
   C. ii and iii  
   D. i, ii and iii  

5. Look at the following Venn diagram:

   ![Venn Diagrams]

   Figure i : \(A \cap B\)  
   Figure ii : \(A \cup B\)  
   Figure iii : \(A \setminus B\)

   Which of the above figure is correct?
   
   A. i and ii  
   B. ii and iii  
   C. i and iii  
   D. i, ii and iii
Based on the information below, answer questions no. (6 Ñ 8):

If \( X = \{a, b\} \), \( Y = \{b, c\} \) and \( Z = \{3, 4\} \), then

6. How many numbers of elements are in \( X \cup Y \cup Z \)?
   A. 2  
   B. 3  
   C. 4  
   D. 5

7. Which one is the correct value of \( P( X \cap Y) \)?
   A. \( \{b, \emptyset\} \)  
   B. \( (b, \emptyset) \)  
   C. \( \{b\}, \emptyset \)  
   D. \( \{b\} \)

8. Which one of the followings indicates \( (X \cap Y) \times Z \)?
   A. \( \{(a, 3), (a, 4)\} \)  
   B. \( \{(b, 3), (b, 4)\} \)  
   C. \( \{(a, 3), (b, 4)\} \)  
   D. \( \{(a, 3), (a, 4), (b, 3), (b, 4), (c, 3), (c, 4)\} \)

**Creative Questions :**

1. A, B and C are three sets where
   
   \[
   A = \{x \in \mathbb{N} : x < 7 \text{ and } x \text{ is an odd number}\} \\
   B = \{x \in \mathbb{N} : x < 7 \text{ and } x \text{ is an even number}\} \\
   C = \{x \in \mathbb{N} : x < 3 \text{ and } x \text{ is a prime number}\}
   \]

   A. Express set A and set B in tabular method.
   B. By determining \( P(A \cap C) \), show that its elements number supports \( 2^n \).
   C. Prove that, \( (A \cap C) \times B = (A \times B) \cap (C \times B) \)

2. Among the students of class nine in your school 55% likes sweet, 65% likes fruits and 30% of them like both of the Tiffin.

   A. Show the above information in Venn diagram with a short description.
   B. Determine the percentage of students does not like both of the Tiffin.
   C. Taking the factor set of number of students like only sweet and number of students like only fruits A and B respectively, express them through Cartesian product. (A is taken as first element in ordered pair).
Chapter II
Real Numbers

To meet up the daily needs of human beings one or two counting numbers were discovered at the out-set of civilization. The gradual development of numbers has created the modern mathematics. So the students of mathematics should have clear idea about numbers.

Calculators and their uses: Mathematical calculations can be done easily with the help of calculators. In an ordinary calculator, there are about 25 keys. Different keys indicate Off, M-in (memory input), MR (memory remind), M Ð (memory minus), M + (Memory plus), %, √, C (cancel), AC/ON, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 and · (decimal), +, ÷, ×, =. The AC key is to be pressed at the beginning of the work. Then the +, ×, ÷ or √ keys are to be pressed according to the need. The result is obtained by pressing the key of =. To keep any number in memory, the M-in key is used. In that case, first the required number is entered by pressing the keys and then the number is stored in the memory by pressing the M-in key. Afterwards the number can be recalled anytime during a calculation by pressing the MR key without pressing the Off or AC key. If any key is pressed by mistake during any mathematical calculation, the result can be cancelled by pressing the C key. The manual for calculator should be read deeply for the use of calculator skillfully.

Example: 15 × 4 = What?
Solution: At first the AC key is pressed. Then the keys for 1 and 5 are pressed to get the number 15. After pressing the × key, the key for 4 is pressed. Finally the = key is pressed to get the result 60.
So, 15 × 4 = 60.

Real Numbers: The numbers 1, 2, 3, ........... etc. are used for counting. To know how many students, fishes, boats etc., we are to tell an exact number such as 1, 2, 3, 4, 5, ........... Such numbers are called counting numbers or natural numbers. The set of these numbers is generally denoted by N, i.e, N = {1, 2, 3, 4, ......}. The smallest member of this set is 1 and there is no greatest member. Beside counting natural numbers are also used for measurement and identification. Such as, 5 kg. of rice, 2 litres of milk or Roll No. 29. The addition of two or more natural numbers is a natural number. But the result of subtraction may not be a natural number. For example, 5 Ð 9 = what? To make the subtraction meaningful, zero and negative integers are introduced. Ð1, Ð2, Ð3, ........ etc. are negative integers. The numbers ..........., Ð3, Ð2, Ð1, 0, 1, 2, 3, ........ etc. are called integers.
The set of all integers is denoted by $\mathbb{Z}$. Thus, $\mathbb{Z} = \{0, 1, -1, 2, -2, 3, -3, \ldots\}$. It is to be noted $\mathbb{N} \subseteq \mathbb{Z}$. There is no smallest or greatest member in the set of integers. The result of addition, subtraction and multiplication operations in the set of integers is again an integer. But if an integer is divided by another integer except zero, the result may not be an integer. For example, $4 \div 5 = \frac{4}{5}$. This type of number is called a rational number. Generally, if $p$ is an integer and $q$ is a non-zero integer, then $\frac{p}{q}$ is a rational number. The set of all rational numbers is denoted by $\mathbb{Q}$. Thus

$$\mathbb{Q} = \left\{ \frac{p}{q} : p, q \in \mathbb{Z} \text{ and } q \neq 0 \right\}.$$ 

Considering $p$ as positive, negative or zero, any rational number can be expressed in the form $\frac{p}{q}$ where $q > 0$. For example, $5 = \frac{5}{1}$, $-8 = \frac{-8}{1}$, $0 = \frac{0}{1}$. It is to be noted that all integers are rational numbers. Hence, $\mathbb{Z} \subseteq \mathbb{Q}$. If $a$ and $b$ be any rational numbers, $a + b$, $a - b$ and $ab$ are rational numbers, $\frac{a}{b}$ is rational, if $b \neq 0$.

There are many numbers which are not rational. They are called irrational numbers. The square root of any natural number which is not a perfect square is an irrational number. So each of $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, $\sqrt{6}$, $\sqrt{7}$, $\sqrt{10}$, ... is an irrational number. $\sqrt{2}$ is an irrational number. An indirect proof of the fact that $\sqrt{2}$ is irrational is given below.

**Proposition** : $\sqrt{2}$ is an irrational number.

**Proof** : $1^2 = 1$, $2^2 = 4$ and $(\sqrt{2})^2 = 2$. Hence, $\sqrt{2}$ is greater than $1$ and smaller than $2$. Therefore, $\sqrt{2}$ is not an integer. If $\sqrt{2}$ is a rational number then let $\sqrt{2} = \frac{p}{q}$, where $p$, $q$ are natural numbers, $q > 1$ and $p$, $q$ are co-prime ($p$, $q$ have no common factors other than $1$). Hence, $2 = \frac{p^2}{q^2}$ or $2q = \frac{p^2}{q}$ [multiplying both sides by $q$]. Evidently $2q$ is an integer. On the other hand, $p^2$ and $q$ do not have common factor as $p$ and $q$ do not have any common factor. Hence, $\frac{p^2}{q}$ is not a natural number. So $\frac{p^2}{q}$ cannot be equal to $2q$.

$\therefore$ $\sqrt{2}$ cannot be equal to any number like $\frac{p}{q}$.

Hence, $\sqrt{2}$ is an irrational number.

**N. B.** (Geometrical explanation of $\sqrt{2}$). If a side of a square has length $1$ unit, then each of its diagonals has length $\sqrt{2}$ unit (Pythagorean theorem).
**Real Number** : The set of real number \( \mathbb{R} \) consists of all rational and irrational numbers. It is to be noted, \( \mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \).

**Number Line**

The real numbers can be represented by points on a straight line. The number line is a line on which a one-to-one correspondence between the real numbers and the points on the line is shown.

![Number Line Diagram](image)

The above figure represents an infinite straight line denoted by \( L \). One point is marked 0 (zero). On the right side of 0, points at successive distances of 1 unit are marked 1, 2, 3, 4 etc. and points at successive distances of 1 unit on the left are marked \( -1, -2, -3, -4 \) etc. The mid-points between 0 and 1, 0 and 2 etc. can be marked \( \frac{1}{2}, \frac{1}{4} \) etc. respectively. In the same way, the points similarly placed on the left side of 0 can be marked \( -\frac{1}{2}, -\frac{1}{4} \) etc. It is to be noted that these are all rational numbers. All the points on number line cannot be covered by the rational numbers. The square root of 2 cannot be completely determined by the division method, but \( \sqrt{2} \) can be geometrically represented on a number-line. Similarly, the irrational numbers \( \sqrt{3}, \sqrt{5}, \sqrt{6}, \ldots \) can be represented on a number-line.

Any number, whether rational or irrational, has a definite representative point on a number-line. On the other hand, any point on a number line is the representative of a definite number, rational or irrational. Thus all the numbers, rational or irrational, have a one-to-one correspondence with the points on the number-line. If \( a, b \) are unequal real numbers, then either \( a > b \) or \( a < b \). In number line, \( a > b \) means that the representative point of \( a \) is on the right of the representative point of \( b \). For example, in figure, \( 3 > 2, 3 > -2, -3 < 2, -3 > -4 \).

**Representation of Real Numbers as Decimals**

Rational numbers can be expressed as finite decimals or recurring decimals. If the factors of \( q \) are only 2 or 5, then the rational number \( \frac{p}{q} \) can be expressed as finite decimals. For example,

\[
\frac{5}{4} = 1.25, \quad \frac{7}{10} = 0.7
\]
If \( q \) has a prime factor other than 2 and 5, then the value of \( \frac{p}{q} \) can be expressed in recurring decimals. For example, \( \frac{5}{111} = \frac{5}{3 \times 37} = 0.045045 \ldots = 0.0\overline{45} \).

Conversely, any finite or recurring decimal is a rational number. A finite decimal number can be expressed as an infinite decimal by placing zeros on the right of the number. Such a number can also be expressed as a recurring decimal. For example, \( 0.\overline{3} = 0.300000 \), \( 0.\overline{29} = 0.29999 \ldots = 0.29 \).

An infinite decimal number which is not recurring is always an irrational number. For example, \( 0.101001000100001000001 \ldots \), \( 0.12112111211112 \ldots \), \( 0.303003000300003 \ldots \) are each an irrational number.

In finding the square roots, cube roots etc., irrational numbers often appear. But in our daily life in trades, we only use rational approximations of irrational numbers. It is to be noted that the irrational numbers and their rational approximations are not same, though we often equate them equal, such as: \( \sqrt{2} = 1.4141 \).

In fact, \( \sqrt{2} = 1.41421356 \ldots \approx 1.414 \).

The symbol \( \approx \) is used to indicate the approximate value of a number.

**Absolute Value:**
If \( a > 0 \), the absolute value of \( a \) is \( a \), if \( a < 0 \), the absolute value of \( a \) is \( -a \) and if \( a = 0 \), the absolute value of \( a \) is zero. The absolute value of \( a \) is denoted by \( |a| \).

i.e. \[ |a| = \begin{cases} a, & \text{if } a > 0 \\ -a, & \text{if } a < 0 \\ 0, & \text{if } a = 0 \end{cases} \]

For example, \( |3| = 3 \), \( |-3| = -(-3) = 3 \), \( |0| = 0 \).

For any number \( a \) and \( b \), \( |ab| = |a| \cdot |b| \). The difference between \( a \) and \( b \) is the absolute value of the subtraction result of one from another, i.e. \( a \sim b = |a \triangle b| = |b \triangle a| \). The sign \( \sim \) indicates the difference between two numbers.

**Distance between numbers:**
In real line, the distance between two corresponding points of two numbers denotes the distance between the numbers. It is observed from the number line
that distance between the numbers 2 and \( \sqrt{2} \) is 4. The distance is found by subtracting the smaller number from the greater number. For example, the distance between \( \sqrt{3} \) and \( \sqrt{27} \) is \( \sqrt{3} - \sqrt{27} = \sqrt{3} + 27 = 24 \), as \( \sqrt{3} > \sqrt{27} \).

**Example:** Find the distance between \( 5 \) and \( -2 \).

**Solution:** As \( 5 \) is positive and \( -2 \) is negative, so \( 5 > -2 \). Hence the distance between \( 5 \) and \( -2 \) is \( 5 - (-2) = 5 + 2 \).

**Some Characteristics of Real Number.**

1. If \( a \in \mathbb{R}, \ b \in \mathbb{R} \), then \( a + b \in \mathbb{R} \) and \( ab \in \mathbb{R} \)
2. If \( a \in \mathbb{R}, \ b \in \mathbb{R} \), then \( a + b = b + a \) and \( ab = ba \).
3. If \( a \in \mathbb{R}, \ b \in \mathbb{R}, \ c \in \mathbb{R} \), then \( (a + b) + c = a + (b + c) \) and \( (ab)c = a(bc) \).
4. There are two particular numbers 0 and 1 where \( 0 \neq 1 \) and \( a + 0 = a \) and \( a.1 = a \).
5. If \( a \in \mathbb{R} \), then \( a + (\sqrt{a}) = 0 \) and if \( a \in \mathbb{R} \), then \( a \neq 0 \), \( a \cdot \frac{1}{a} = 1 \).
6. If \( a,b, \ c \in \mathbb{R} \), then \( a(b + c) = ab + ac \).
7. If \( a, \ b \in \mathbb{R} \), one and only one condition of the following is applicable:
   - \( a = b \), \( a > b \), \( a < b \).
8. If \( a, \ b, \ c \in \mathbb{R} \) and \( a < b \), then \( a + c < b + c \).
9. If \( a, \ b, \ c \in \mathbb{R} \) and \( a < b \), then \( ac < bc \) when \( c > 0 \) and \( ac > bc \) when \( c < 0 \).

**Example 1.** Solve : \( |x| = 2 \).

**Solution:** If \( x \) is non-negative, \( |x| = x = 2 \).
   - If \( x \) is negative, \( |x| = -x = 2 \) \( \Rightarrow \ x = \sqrt{2} \).
   - Thus, \( x = 2 \) or \( x = \sqrt{2} \).

**Remark:** In a number line, only 2 and \( \sqrt{2} \) satisfy the equation. Hence we can say that, the solution set of the equation \( |x| = 2 \) is \( S = \{ 2, \sqrt{2} \} \).

**Example 2.** Solve and show the solution set in the number line : \( |x| < 3 \).

**Solution:** If \( x \) is non-negative \( |x| = x < 3 \) i.e., the value of \( x \) may be any non-negative number less than 3. Thus, in this case, \( 0 \leq x < 3 \).
   - Again, if \( x \) is negative, \( |x| = -x < 3 \) or \( x > -3 \) [multiplying both sides by \( \sqrt{1} \)] i.e., the value of \( x \) is any negative number greater than \( -3 \). Thus in this case \( -3 < x < 0 \).
   - \( \Rightarrow \ -3 < x < 0 \) or \( 0 \leq x < 3 \), i.e., \( -3 < x < 3 \).
Thus the solution set is $S = \{ x \in \mathbb{R} : \exists 3 < x < 3 \}$. 

It is to be noted that the circles are drawn at 3 and $\exists 3$ and they are not filled to indicate that 3 and $\exists 3$ are excluded from the solution set.

**Remark:** In case of inequality, multiplication or division by a negative number the sign of inequality gets reverse.

**Example 3.** Find a rational number and an irrational numbers between $a$ and $b$ where 
$a = 0.202002000200002\ldots\ldots\ldots\ldots\ldots\ldots$
$b = 0.2002000200002000002\ldots\ldots\ldots\ldots\ldots\ldots$

**Solution:** $a$ and $b$ are infinite and non-recurring decimal numbers i.e. irrational numbers.
Let us consider the rational number $c = 0.201$. 
We note that the third digit on the right of decimal point of $a$ is 2, 
the third digit on the right of decimal point of $b$ is 0, 
the third digit on the right of the decimal point of $c$ is 1 and $0 < 1 < 2$. 
Hence, $a$ is greater than $c$ and $c$ is greater than $b$, that is, $a > c > b$. 
Again, let us consider the number $d = 0.201002000200002\ldots\ldots\ldots\ldots\ldots\ldots$ it is an irrational number. 
We note that the third digit on the right side of $a$ is 2, 
the third digit on the right side of the decimal of $b$ is 0, 
the third digit on the right side of $d$ is 1 and $0 < 1 < 2$. Hence, $a > d > b$. 
Here $d$ is infinite and non-recurring. Hence $d$ is an irrational number.

**N.B.** There are infinitely many rational numbers and irrational numbers between any two real numbers.

**Example 4.** Find two irrational numbers between the numbers 2 and $2\sqrt{5}$.

**Solution:** Let us consider two numbers $a$ and $b$, where 
$a = 2.10100100010000\ldots\ldots\ldots\ldots\ldots\ldots$
and $b = 2.202002000200002\ldots\ldots\ldots\ldots\ldots\ldots$

Clearly, $2 < 2.1010010001\ldots\ldots\ldots\ldots\ldots\ldots < 2\sqrt{5}$ and $2 < 2.202002000200002\ldots\ldots\ldots\ldots\ldots\ldots < 2\sqrt{5}$. 
a and $b$ lie between 2 and $2\sqrt{5}$ and $a$ and $b$ are both irrational numbers. 
\[ \therefore \] a and $b$ are irrational numbers lying between 2 and $2\sqrt{5}$. 


Example 5. Show that, \( \frac{2}{\sqrt{5} \cdot \sqrt{3}} = \sqrt{5} + \sqrt{3} \) and find the approximate value upto 3 decimal places.

**Solution:** \( \frac{2}{\sqrt{5} \cdot \sqrt{3}} = \frac{2(\sqrt{5} + \sqrt{3})}{(\sqrt{5} \cdot \sqrt{3})(\sqrt{5} + \sqrt{3})} = \frac{2(\sqrt{5} + \sqrt{3})}{5 \cdot 3} = \frac{2(\sqrt{5} + \sqrt{3})}{15} \)

\[ = \sqrt{5} + \sqrt{3} = 2\sqrt{23606} + 1\sqrt{73205} = 3\sqrt{96811} = 3\sqrt{968} \]

[The symbol \( \approx \) is used to indicate approximate values].

Example 6. Solve : \( |x + 3| < 5 \) and indicate the solution set on a number line.

**Solution:** If \( x + 3 \geq 0 \), i.e., \( x > -3 \), then the inequality becomes,
\[ x + 3 < 5 \]
\[ \text{or, } x < 5 \]
\[ \text{or, } x < 2 \]
\[ \therefore \text{ in this case, } -3 \leq x \text{ and } x < 2, \]

Again, if \( x + 3 < 0 \), i.e., \( x < -3 \), then the inequality becomes,
\[ -x - 3 < 5 \]
\[ \text{or, } x + 3 > -5 \text{ (Multiplying both the sides by -1)} \]
\[ \text{or, } x > -5 \]
\[ \text{or, } x > -8 \]
\[ \therefore \text{ in this case, } -8 < x \text{ and } x < -3, \text{ so that } -8 < x < -3. \]

Hence, \( -8 < x < -3 \) or \( -3 < x < 2 \)
\[ \therefore \text{ The required solutions are } -8 < x < 2 \text{ and the solution set is } S = \{ x \in \mathbb{R} : -8 < x < 2 \}. \]

The solution set \( S \) is indicated on a number line:

![Number Line](image)


e8 e7 e6 e5 e4 e3 e2 e1 0 1 2

Example 7. Show that, if the square of an odd natural number is divided by 8 then in each case the remainder will be 1.

**Solution:** If \( n \) is an odd natural number,
\[ n = 2x + 1 \text{ where } x \in \mathbb{N}, \text{ in this case} \]
\[ n^2 = (2x + 1)^2 = 4x^2 + 4x + 1 = 4x(x + 1) + 1 \]
If \( n = 1, n^2 = 1, \) the remainder is 1 when divided by 8.

Since, \( x \) and \( x + 1 \) are two consecutive natural numbers, one of them must be even number. Hence \( x(x + 1) \) is divided by 2, as result \( 4x(x + 1) \) is divided by \( 4 \times 2 = 8 \).

Therefore, when the square of an odd natural number is divided by 8 then in each case the remainder will be 1.
Exercise 2

1. Determine the approximate value upto two decimal places and show, in the number line :
   (i) \( \sqrt{17} \)  (ii) \( \sqrt{18} \)  (iii) \( \frac{\sqrt{3}}{2} \)  (iv) \( 1 + \sqrt{2} \)  (v) \( \sqrt{2} \) \( \approx 1.41 \) 

2. Solve and show the solution set in a number line :
   (i) \( |x| \leq 4 \)  (ii) \( 1 < |x| < 2 \)  (iii) \( |x| = \sqrt{2} \)  (iv) \( \frac{|x|}{2} = 5 \)

3. Determine the distance :
   (i) \( \mathbb{D} 2 \) and \( \mathbb{D} 3 \)  (ii) \( \mathbb{D} 3 \) and \( \mathbb{D} 4 \)  (iii) \( \mathbb{D} 5 \) and \( \mathbb{D} \frac{3}{4} \)

4. Solve : (i) \( |x| \mathbb{D} \frac{5}{9} < 4 \)  (ii) \( |x| \mathbb{D} \frac{5}{9} = 4 \)  (iii) \( |x| \mathbb{D} \frac{5}{9} > 4 \)

5. Find two irrational numbers between 0.1 and 0.12.

6. Find the approximate values of \( \sqrt{2} \) and \( \sqrt{3} \) upto 4 decimal places using calculator and find two irrational numbers between them.

7. Find an irrational number between 0.1 and 0.1101.

8. Find the solution sets : (i) \( |3x + 2| < 7 \)  (ii) \( \left| \frac{x + 2}{x + 5} \right| = 3 \) \( [x \neq \mathbb{D} 5] \)

9. Find the value upto 3 decimal places of \( \frac{1}{\sqrt{2} + \sqrt{3}} \)

10. Find the value upto 4 decimal places of \( \frac{1}{\sqrt{2} + \sqrt{3}} \)

11. Find the value upto 4 decimal places :
    (i) \( \frac{2 + \sqrt{5}}{3 \sqrt{5}} \)  (ii) \( \frac{\sqrt{3} \mathbb{D} \sqrt{2}}{\sqrt{3} + \sqrt{2}} \)
Multiple Choice Questions (MCQ) :

1. Which of the following relations is correct in terms of set?
   A. \( \mathbb{N} \subset \mathbb{Q} \subset \mathbb{Z} \subset \mathbb{R} \)       
   B. \( \mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \)       
   C. \( \mathbb{Z} \subset \mathbb{N} \subset \mathbb{Q} \subset \mathbb{R} \)       
   D. \( \mathbb{Z} \subset \mathbb{N} \subset \mathbb{R} \subset \mathbb{Q} \)

2. If \( P = \{3\} \), then what is the appropriate value of \(|P|\)?
   A. \( \{3\} \)       
   B. \( 0 \)       
   C. \( \pm 3 \)       
   D. \( 3 \)

3. Which of the following is the number line expression of set \( S = \{x \in \mathbb{R} : -1 < x \leq 2\} \)?
   A.       
   B.       
   C.       
   D.       

4. See the following sentences : 
   i. \( 0 \) is a natural number 
   ii. \( \sqrt{8} \) is a irrational number 
   iii. All the natural numbers are real number 

Based on the above information, which one of the following is correct?
   A. i and ii       
   B. ii and iii       
   C. i and iii       
   D. i, ii and iii 

Based on the following information, answer questions no. (5 Ñ 7) :
\( f(x) = x^2 - 2ax + (a + b)(a - b) \)

5. If \( x = a \), then which one of the following is the appropriate value of \(|f(x)|\) ?
   A. \( b \)       
   B. \( \mathbb{D} b \)       
   C. \( b^2 \)       
   D. \( \mathbb{D} b^2 \)
6. If \( f(x) = 0 \), then which of the following is the appropriate solution set?
   A. \( \{ x \in \mathbb{R} : x = \overline{a b} \text{ or } x + a + b \} \)
   B. \( \{ x \in \mathbb{R} : x = a + b \text{ or } x = a \overline{+ b} \} \)
   C. \( \{ x \in \mathbb{R} : x = a \overline{+ b} \text{ or } x = a \overline{+ b} \} \)
   D. \( \{ x \in \mathbb{R} : x = a \overline{+ b} \text{ or } x = a + b \} \)

7. If \( a = 0\overline{1020} \) and \( b = 0\overline{1101} \), then which one of the following is the correct irrational number between \( a \) and \( b \)?
   A. \( 0\overline{101020020002} \)
   B. \( 0\overline{101010010001} \)
   C. \( 0\overline{102010010001} \)
   D. \( 0\overline{1101202002} \)

Creative Questions:

1. Deep and Deepa scores \( x \) and 65 in mathematics in their annual examination. The difference of their scores is not greater than 3 and is not less than 2.
   A. Show the above information as inequality.
   B. Solve the inequality.
   C. Show the solution set on a number line and find an irrational number lies between 2 and 3.
**Chapter III**  
**Algebraic Expression**

**Algebraic Expression**: Operations of addition, subtraction, multiplication, division etc. are carried out by numbers of fixed values (constant) in arithmetic. But in algebra, besides the numbers of fixed values letters a, b, c, x, y, z, α, β etc. of alphabets are used to represent numbers of variable values. In arithmetic, only the positive numbers are used. Generally arithmetical calculations are used in daily life. In algebra, all numbers, positive or negative including zero are used. Algebra may be considered as the generalisation of arithmetic. The sign $\times$ is used for multiplication in arithmetic. But in algebra it is not generally used. One reason may be that the multiplication sign $\times$ and English letter x may cause confusion. In algebra, ab stands for $a \times b$ (or a.b). Hence if $a = 2$, $b = 3$, then $ab = 2.3 = 6$. But in arithmetic 23 stands for $2.10 + 3$ (in 10-based or decimal numeration). If the degree of numerator is less than that of denominator in an algebraic fraction then the fraction is called a proper fraction in algebra. For example, $\frac{x^2 + x + 2}{x^3 + 2x}$ is proper fraction. If the numerator is not of lower degree than the denominator, then the fraction is improper one.

For example, $\frac{x^3 + 1}{x^2 + x + 1}$ and $\frac{x^3 + x + 1}{x^3 \div x}$ are both improper fractions. An improper fraction can be expressed as the sum of a polynomial (integral part) and a proper fraction by the method of division.

For example, $\frac{x^2 + 3}{x \div 1} = (x +1) + \frac{4}{x \div 1}$

**Variable**: A symbol that represents any element of a fixed set is called a variable. For example, given $A = \{x \in R : 1 \leq x \leq 20\}$, x is a variable and its value is any number from 1 to 20.

**Power**: $a^n$ is the nth power of a.

**Formula**: A formula is an equation involving one or more variables where the equation is satisfied by all values of the related variables. Or any general rule expressed by symbols is also a formula.

**Formula**: $(p + x) (q + x) = pq + (p + q)x + x^2$
Proof: \[(p + x) (q + x) = p(q + x) + x(q + x) = pq + px + qx + x^2 = pq + x(p + q) + x^2\]

Corollary: (i) \((a + b)^2 = (a + b)(a + b) = a.a + (a + a)b + b^2 = a^2 + 2ab + b^2\)

(ii) \((a \cdot b)^2 = \{(a + (\cdot b)) \{a + (\cdot b)\} = a.a + (a + a)(\cdot b) + (\cdot b)(\cdot b) = a^2 \cdot 2ab + b^2\)

(iii) \[a^2 + b^2 = (a + b)^2 \cdot 2ab = (a \cdot b)^2 + 2ab = \frac{(a + b)^2 + (a \cdot b)^2}{2}\]

(iv) \((a + b)^2 = (a \cdot b)^2 + 4ab\)

(v) \((a \cdot b)^2 = (a + b)^2 \cdot 4ab\)

(vi) \[4ab = (a + b)^2 \cdot (a \cdot b)^2\]

(vii) \[ab = \left(\frac{a + b}{2}\right)^2 \cdot \left(\frac{a \cdot b}{2}\right)^2\]

(viii) \[(a + b) (a \cdot b) = a^2 \cdot b^2\]

Extension of Formula for square:

\[(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca.\]

Corollary: \[a^2 + b^2 + c^2 = (a + b + c)^2 \cdot 2(ab + bc + ca) = 2(ab + bc + ca) = (a + b + c)^2 \cdot (a^2 + b^2 + c^2).\]

Example 1. What is the square of \((3a \cdot 2x)\)?

Solution: \[(3a \cdot 2x)^2 = (3a)^2 \cdot 2.3a.2x + (2x)^2 = 9a^2 \cdot 12ax + 4x^2\]

Example 2. Simplify: \((3x + 2y)^2 + 2(3x + 2y)(3x \cdot 2y) + (3x \cdot 2y)^2\)

Solution: Here, let \(3x + 2y = a\) and \(3x \cdot 2y = b\), then

given expression \[= a^2 + 2ab + b^2 = (a + b)^2 = \{(3x + 2y) + (3x \cdot 2y)^2\}^2 [putting the values of a and b] = (3x + 2y + 3x \cdot 2y)^2 = (6x)^2 = 36x^2.\]
Example 3. If \( a + b = 7 \) and \( ab = 12 \), then what is the value of \( a - b \)?

Solution:

\[
(a - b)^2 = (a + b)^2 - 4ab = 7^2 - 4 \cdot 12 = 49 - 48 = 1
\]

\[\therefore\ a - b = \pm 1.\]

Example 4. If \( x - y = 1 \) and \( xy = 56 \), then what is the value of \( x + y \)?

Solution:

\[
(x + y)^2 = (x - y)^2 + 4xy = 1^2 + 4 \cdot 56 = 1 + 224 = 225
\]

\[\therefore\ x + y = \pm \sqrt{225} = \pm 15.\]

Example 5. If \( x + \frac{1}{x} = \sqrt{2} \), then show that, \( x^2 + \frac{1}{x^2} = 0 \)

Solution:

\[
x^2 + \frac{1}{x^2} = \left( x + \frac{1}{x} \right)^2 = 2 \cdot \frac{1}{x} = (\sqrt{2})^2 = 2 = 2 \cdot 2 = 0
\]

Example 6. If \( x + \frac{1}{x} = 5 \) then, find the value of \( \frac{x}{x^2 + x + 1} \) [where \( x \neq 0 \)]

Solution:

Here, \( x + \frac{1}{x} = 5 \) and \( x \neq 0.\)

\[
\therefore\ \frac{x}{x^2 + x + 1} = \frac{x}{x + 1 + \frac{1}{x}} = \frac{1}{x + 1 + \frac{1}{x}} = \frac{1}{x + \frac{1}{x} + 1} = \frac{1}{5 + 1} = \frac{1}{6}
\]

Example 7. Show that, \((a + 2b) (3a + 2c)\) is equal to the difference of two perfect squares.

Solution:

\[
(a + 2b) (3a + 2c) = \left( a + 2b + 3a + 2c \right)^2 \cdot \left( a + 2b + 3a + 2c \right)^2
\]

\[
= \left( 4a + 2b + 2c \right)^2 \cdot \left( 2a + 2b + 2c \right)^2
\]

\[
= \left( 2(2a + b + c) \right)^2 \cdot \left( 2(2a + b + c) \right)^2
\]

\[
= (2a + b + c)^2 \cdot (2a + b + c)^2
\]

Example 8. If \( a + b + c = 9 \) and \( a^2 + b^2 + c^2 = 29 \), then what is the value of \( ab + bc + ca \)?

Solution:

Here, \( 2(ab + bc + ca) = (a + b + c)^2 \cdot (a^2 + b^2 + c^2) \)

\[\therefore\ ab + bc + ca = \frac{52}{2} = 26.\]
**Example 9.** If \( x + y + z = 2 \) and \( xy + yz + zx = 1 \), then what is the value of \((x + y)^2 + (y + z)^2 + (z + x)^2\)?

**Solution:**

\[
(x + y)^2 + (y + z)^2 + (z + x)^2
= x^2 + 2xy + y^2 + y^2 + 2yz + z^2 + z^2 + 2zx + x^2
= (x^2 + y^2 + z^2 + 2xy + 2yz + 2zx) + x^2 + y^2 + z^2
= (x + y + z)^2 + (x + y + z)^2 - 2(xy + yz + zx)
= 2^2 + 2^2 - 2.1 = 4 + 4 - 2 = 6.
\]

**Example 10.** If \( \frac{6}{x} = 1 \), then what is the value of \( \frac{6}{x^2 + x + 1} \)?

**Solution:**

\[
x - \frac{6}{x} = 1 \quad \text{or,} \quad \frac{x^2 - 6}{x} = 1 \quad \text{or,} \quad x^2 - 6 = x
\]

\[
\text{or,} \quad x^2 \cdot x \cdot 6 = 0 \quad \text{or,} \quad (x \cdot 3) (x + 2) = 0
\]

\[
\therefore \quad x = 3 \quad \text{or,} \quad x = 2
\]

Hence, \( x = 3 \) or, \( x = 2 \)

when, \( x = 3 \),
\[
\frac{6}{x^2 + x + 1} = \frac{6}{3^2 + 3 + 1} = \frac{6}{13}
\]

when, \( x = 2 \),
\[
\frac{6}{x^2 + x + 1} = \frac{6}{(2)^2 + 2 + 1} = \frac{6}{3} = 2
\]

**Answer:** 2 or, \( \frac{6}{13} \)

**Exercise 3.1**

1. Using a formula, find the square of:
   (i) \( a + 3b \)  
   (ii) \( ab \cdot c \)  
   (iii) \( x^2 + \frac{2}{y^2} \)  
   (iv) \( 3p + 4q \cdot 5r \)
   (v) \( \frac{a}{2} + \frac{2}{b} \cdot \frac{1}{c} \)  
   (vi) 996  
   (vii) \( ax \cdot by \cdot cz \)

2. Simplify:
   (i) \( (4x + 7y \cdot 3z)^2 + 2(4x + 7y \cdot 3z) (7y \cdot 4x + 3z) + (7y \cdot 4x + 3z)^2 \)
   (ii) \( (a \cdot b + c)^2 \cdot 2(b + c \cdot a) (a \cdot b + c) + (b + c \cdot a) \)
   (iii) \( \frac{8ú625 \times 8ú625 \cdot 2 \times 8ú625 \times 6ú375 + 6ú375 \times 6ú375}{8ú625 \cdot 6ú375} \)
3. Find the value of $64x^2 + 96xy + 37y^2$, when $x = \frac{1}{8}$ and $y = 1$.

4. When $x \div \frac{1}{x} = a$, what is the value of $x^2 + \frac{1}{x^2}$?

5. When $a + b = 7p$ and $ab = 12p^2$, what is the value of $a \div b$?

6. If $x \div y = 2$ and $xy = 3$, what is the value of $x + y$?

7. If $x + \frac{1}{x} = 2$, what is the value of $x^4 + \frac{1}{x^4}$?

8. If $x + \frac{1}{x} = 4$, what is the value of $\frac{1}{x^2} \div 3x + 1$?

9. If $x + y = 12$ and $x \div y = 2$, (i) what is the value of $x^2 + y^2$?
   (ii) What is the value of $xy$?

10. If $a + b = \sqrt{3}$ and $a \div b = \sqrt{2}$, then prove that, $8ab (a^2 + b^2) = 5$.

11. Express 45 as the difference of two squares.

12. If $x + y + z = 15$ and $x^2 + y^2 + z^2 = 83$, then what is the value of $xy + yz + zx$?

13. If $x + y + z = p$ and $xy + yz + zx = q$, then what is the value of $(x + y)^2 + (y + z)^2 + (z + x)^2$?

14. If $a + b + c = 10$ and $a^2 + b^2 + c^2 = 38$, then what is the value of $(a \div b)^2 + (b \div c)^2 + (c \div a)^2$?

15. If $x \div \frac{1}{x} = p$, then find the value of $\frac{c}{x(x \div p)}$.

16. Show that,
   \[
   \left( \frac{x + y}{2} \right)^2 \div \left( \frac{x \div y}{2} \right)^2 = \left( \frac{x^2 + y^2}{2} \right)^2 \div \left( \frac{x^2 \div y^2}{2} \right)^2.
   \]

17. Show that, $(3a + 4b)(5a + 2c)$ is equal to the difference of two perfect squares.

18. If $p = 3 + \frac{1}{p}$, then prove that, $p^4 = 119 \div \frac{1}{p^4}$.
19. If \( x = \sqrt{3} + \sqrt{2} \), then find the value of \( x^2 + \frac{1}{x^2} \).

20. If \( x = b \pm c \), \( y = c \pm a \), \( z = a \pm b \), find the value of \( 2x^2 + y^2 + z^2 + 2xz \).

21. Express \( x^2 + 8x \pm 20 \) as the difference of two squares.

**Formulae of Cubes**

**Formula:**  
\[(p + x)(q + x)(r + x) = pqr + (pq + qr + rp)x + (p + q + r)x^2 + x^3\]

**Proof:**  
We know that, \((p + x)(q + x) = pq + (p + q)x + x^2\)  
Hence, \((p + x)(q + x)(r + x) = (pq + (p + q)x + x^2)(r + x)\)  
\[= pqr + (p + q)xr + x^2r + pqx + (p + q)x^2 + x^3\]

**Formula:**  
\[(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3 = a^3 + b^3 + 3ab(a + b)\]

**Proof:**  
\[(a + b)^3 = (a + b)(a + b)(a + b)\]
\[= a.a.a + (a.a. + a.a + a.a)b + (a + a + a)b^2 + b^3\]
[putting a for p, q, r and b for x in the above formula]  
\[= a^3 + 3a^2b + 3ab^2 + b^3\]
\[= a^3 + b^3 + 3a^2b + 3ab^2\]
\[= a^3 + b^3 + 3ab(a + b)\]

**Alternative Proof:**
\[(a + b)^3 = (a + b)(a + b)^2 = (a + b)(a^2 + 2ab + b^2)\]
\[= a(a^2 + 2ab + b^2) + b(a^2 + 2ab + b^2)\]
\[= a^3 + 2a^2b + ab^2 + a^2b + 2ab^2 + b^3\]
\[= a^3 + 3a^2b + 3ab^2 + b^3\]

**Formula:**  
\[(a \pm b)^3 = a^3 \pm 3a^2b + 3ab^2 \pm b^3 = a^3 \pm b^3 \pm 3ab(a \pm b)\]

**Proof:**  
\[(a \pm b)^3 = (a \pm (\pm b))^3 = (a + (\mp b)) \{a + (\mp b)\} \{a + (\mp b)\}\]
\[= a.a.a + (a.a + a.a + a.a)(\mp b) + (a + a + a)(\mp b)^2 + (\mp b)^3\]
\[= a^3 \mp 3a^2b + 3ab^2 \mp b^3\]
\[= a^3 \mp b^3 \mp 3a^2b + 3ab^2\]
\[= a^3 \mp b^3 \mp 3ab(a \mp b)\]
**Formula:** \( a^3 + b^3 = (a + b)(a^2 - ab + b^2) \)

**Proof:**
\[
a^3 + b^3 = a^3 + b^3 + 3ab(a + b) - 3ab(a + b)
= (a + b)^3 - 3ab(a + b)
= (a + b)((a + b)^2 - 3ab)
= (a + b)(a^2 + 2ab + b^2 - 3ab)
= (a + b)(a^2 - ab + b^2)
\]

**Formula:** \( a^3 - b^3 = (a - b)(a^2 + ab + b^2) \)

**Proof:**
\[
a^3 - b^3 = a^3 - b^3 + 3ab(a - b) + 3ab(a - b)
= (a - b)^3 + 3ab(a - b)
= (a - b)((a - b)^2 + 3ab)
= (a - b)(a^2 - 2ab + b^2 + 3ab)
= (a - b)(a^2 + ab + b^2)
\]

**Alternative proof:**
\[
a^3 - b^3 = a^3 + (-(b))^3
= (a - b)(a^2 - a(b) + (b)^2)
= (a - b)(a^2 + ab + b^2)
\]

**Example 1.** Multiply by means of a formula: \((3 + x)(4 + x)(7 + x)\)

**Solution:**
\[
(3 + x)(4 + x)(7 + x)
= 3.4.7 + (3 + 4 + 7)x + 3 + 4 + 7)x^2 + x^3
= 84 + 61x + 14x^2 + x^3
\]

**Example 2.** Find the cube of \((a + 2b)\).

**Solution:**
\[
(a + 2b)^3 = a^3 + 3a^2.2b + 3a.(2b)^2 + (2b)^3
= a^3 + 6a^2b + 12ab^2 + 8b^3
\]

**Example 3.** Find the cube of \(\left(\frac{1}{p}\right)\).

**Solution:**
\[
\left(\frac{1}{p}\right)^3 = p^3 \cdot \frac{1}{p} + 3p\left(\frac{1}{p}\right)^2 \cdot \frac{1}{p}
= p^3 \cdot \frac{1}{p} + \frac{3}{p^3}
\]
**Example 4.** Simplify : \((2x + 3y - 4z)^3 + (2x - 3y + 4z)^3 + 12x(4x^2 - (3y - 4z)^2)\)

**Solution:** Let, \(a = 2x + 3y - 4z\) and \(b = 2x - 3y + 4z\)

Then, \(a + b = 4x\)

\[\therefore \text{ Given expression } = (2x + 3y - 4z)^3 + (2x - 3y + 4z)^3 + 3(4x)(2x + 3y - 4z)(2x - 3y + 4z) \]

\[= a^3 + b^3 + 3(a + b)ab \]

\[= a^3 + b^3 + 3ab(a + b) \]

\[= (a + b)^3 = (4x)^3 = 64x^3 \]

**Example 5.** If \(x = 6\), then what is the value of \(8x^3 - 72x^2 + 216x - 216\)?

**Solution:**

\[8x^3 - 72x^2 + 216x - 216 = (2x)^3 - 3(2x)^2.6 + 3.2x.6^2 - 6^3 \]

\[= (2x - 6)^3 = (2.6 - 6)^3 \]

\[\because \ x = 6 \]

\[= (12 - 6)^3 = (6)^3 = 216. \]

**Example 6.** If \(x + y + z = 0\), prove that, \(x^3 + y^3 + z^3 = 3xyz\)

**Solution:** Given \(x + y + z = 0\)

or, \(x + y = -z\)

Hence, \((x + y)^3 = (z)^3\)

or, \(x^3 + y^3 + 3xy(x + y) = z^3\)

or, \(x^3 + y^3 + 3xy(z) = z^3 \) \[\therefore x + y = z\]

or, \(x^3 + y^3 + z^3 = 3xyz\)

**Example 7.** If \(a + \frac{1}{a} = 3\), prove that, \(a^3 + \frac{1}{a^3} = 0.\)

**Solution:**

\[a^3 + \frac{1}{a^3} = \left(a + \frac{1}{a}\right)^3 \cdot 3a\frac{1}{a} \left(a + \frac{1}{a}\right) \]

\[= \left(a + \frac{1}{a}\right)^2 \left(a + \frac{1}{a}\right) \cdot 3 \left(a + \frac{1}{a}\right) \]

\[= 3 \left(a + \frac{1}{a}\right) \cdot 3 \left(a + \frac{1}{a}\right) \]

\[\therefore \left(a + \frac{1}{a}\right)^2 = 3 \]

\[= 0. \]
Example 8. If \( x + y = 2 \), \( x^2 + y^2 = 4 \), then what is the value of \( x^3 + y^3 \)?

Solution:

\[ x + y = 2 \quad \text{or,} \quad x^2 + 2xy + y^2 = 4 \]
\[ \text{or,} \quad 4 + 2xy = 4 \quad \text{[putting} \ x^2 + y^2 = 4 \text{]} \]
\[ \text{or,} \quad 2xy = 4 - 4 = 0 \]
\[ xy = 0 \]

\[ \therefore \quad x^3 + y^3 = (x + y)^3 - 3xy(x + y) = 2^3 - 3 \times 0 \times 2 = 8 \]

Example 9. If \( x + y = a \), \( x^2 + y^2 = b^2 \) and \( x^3 + y^3 = c^3 \), then show that,

\[ a^3 + 2c^3 = 3ab^2. \]

Solution:

\[ a^3 + 2c^3 = (x + y)^3 + 2(x^3 + y^3) \]
\[ = x^3 + y^3 + 3xy(x + y) + 2(x^3 + y^3) \]
\[ = 3(x^3 + y^3) + 3xy(x + y) \]
\[ = 3((x^3 + y^3) + xy(x + y)) \]
\[ = 3((x + y)(x^2 - xy + y^2) + xy(x + y)) \]
\[ = 3(x + y)(x^2 + y^2) \]
\[ = 3ab^2 \quad \text{[} \because \ x + y = a, \ x^2 + y^2 = b^2 \text{]} \]

Example 10. If \( x - y = 8 \) and \( xy = 65 \), then what is the value of

\[ x^3 - y^3 - 16(x - y)^2 \]

Solution:

\[ x^3 - y^3 - 16(x - y)^2 \]
\[ = (x - y)^3 + 3xy(x - y) - 16(x - y)^2 \]
\[ = 8^3 + 3 \times 65 \times 8 - 16 \times 8^2 \]
\[ = 8(64 + 260 - 128) \]
\[ = 8 \times 131 = 1048. \]

Example 11. Simplify:

\[ (a \cdot b)(a^2 + ab + b^2) + (b \cdot c)(b^2 + bc + c^2) + (c \cdot a)(c^2 + ca + a^2) \]

Solution:

\[ (a \cdot b)(a^2 + ab + b^2) + (b \cdot c)(b^2 + bc + c^2) + (c \cdot a)(c^2 + ca + a^2) \]
\[ = a^3 \cdot b^3 + b^3 \cdot c^3 + c^3 \cdot a^3 = 0. \]
**Exercise 3.2**

1. Multiply: (i) \((a + x)(b + x)(c + x)\)  
   (ii) \((4 + x)(3 + x)(2 + x)\)

2. Find the cube of: (i) \(3x - 4y\)  
   (ii) \(a + b + c\)  
   (iii) 403

3. Simplify:
   (i) \((x + y)(x^2 + xy + y^2) + (y + z)(y^2 + yz + z^2) + (z + x)(z^2 + zx + x^2)\)
   (ii) \((4a + 3b)^3 = 3(4a + 3b)(2a + 3b) + 3(4a + 3b)(2a + 3b) + (2a + 3b)^3\)
   (iii) \((a + b + c)^3 = (a + b + c)(a^2 + b + c + 3abc)\)

4. If \(x = 19\) and \(y = 12\), then find the value of \(8x^3 + 36x^2y + 54xy^2 + 27y^3\).

5. If \(a + b = 3\) and \(ab = 2\), then find the value of \(a^3 + b^3\).

6. If \(a^3 - b^3 = 513\) and \(a + b = 3\), then what is the value of \(ab\)?

7. If \(a + b = c\), then show that, \(a^3 + b^3 + 3abc = c^3\).

8. If \(x + \frac{1}{x} = \sqrt{3}\), then what is the value of \(x^3 + \frac{1}{x^3}\) ?

9. If \(a + b = 5\) and \(ab = 36\), then what is the value of \(a^3 + b^3\)?

10. If \(a + b = m\), \(a^2 + b^2 = n\) and \(a^3 + b^3 = p^3\), then show that, \(m^3 + 2p^3 = 3mn\).

11. If \(x + y = 5\) and \(xy = 6\), then find the value of \(x^3 + y^3 + 4(x + y)^2\).

12. If \(2x + \frac{1}{3x} = 5\), then find the value of \(4x^2 + \frac{1}{9x^2}\) and \(8x^3 + \frac{1}{27x^3}\).

13. If \(\frac{a}{b} + \frac{b}{a} = 6\), then find the value of \(\frac{a^2}{b^2} + \frac{b^2}{a^2}\).

14. If \(x = \sqrt{3} + \sqrt{2}\), then find the value of \(x^3 + \frac{1}{x^3}\).

15. If \(2x + \frac{2}{x} = 3\), then prove that, \(8 \left( x^3 + \frac{1}{x^3} \right) = 63\).
Factors

If an expression is equal to the product of two or more expressions then each of the latter expressions is called a factor of the former. Finding all probable factors of any algebraic expression and then expressing the expression as a product of these factors is called factorisation or resolution into factors. Any expression involving fraction may be resolved into factors in different ways.

For example, 
\[ a^3 + \frac{1}{8} = a^3 + \frac{1}{2^3} = \left( a + \frac{1}{2} \right) \left( a^2 \frac{a}{2} + \frac{1}{4} \right) \]

Again, 
\[ a^3 + \frac{1}{8} = \frac{1}{8} (8a^3 + 1) = \frac{1}{8} \{(2a)^3 + 1^3\} = \frac{1}{8} (2a + 1)(4a^2 - 2a +1) \]

The expression in algebra may consist of one or more terms and as such the factors may also contain one or more terms. In resolving any expression into factors, commutative, associative and distributive laws of multiplication are used.

According to the distributive law of multiplication, 
\[ ka + kb + kc = k(a + b + c) \]

**Example 12.** Resolve into factors : \( a^2b^2m^2 + a^2b^2n^2 + a^2b^2p^2 \).

**Solution :** 
\[ a^2b^2m^2 + a^2b^2n^2 + a^2b^2p^2 = a^2b^2 (m^2 + n^2 + p^2) \]

[N.B. Here it is unnecessary to write \( a^2b^2 \) as \( a \cdot a \cdot b \cdot b \)]

The formula \( a^2 \mp b^2 = (a + b)(a \mp b) \) plays an important role in factorisation.

**Example 13.** Resolve into factors : \( 2x^2 \mp 8y^2 \).

**Solution :** 
\[ 2x^2 \mp 8y^2 = 2(x^2 \mp 4y^2) = 2\{x^2 \mp (2y)^2\} \]
\[ = 2(x + 2y)(x - 2y). \]

**Example 14.** Resolve into factors : \( x^4 \mp 6x^2y^2 + y^4 \).

**Solution :** 
\[ x^4 \mp 6x^2y^2 + y^4 = (x^2)^2 \mp 2x^2 \cdot y^2 + (y^2)^2 \]
\[ = (x^2 \mp y^2)^2 \mp (2xy)^2 \]
\[ = (x^2 \mp y^2)^2 \mp (2xy)(x^2 \mp y^2) \]
\[ = (x^2 + 2xy - y^2)(x^2 - 2xy + y^2) \]
The use of the formulae \(a^3 + b^3 = (a + b)(a^2 - ab + b^2)\) and
\(a^3 - b^3 = (a - b)(a^2 + ab + b^2)\) in resolving into factors is shown below.

**Example 15.** Resolve into factors : \(x^4 + 27x\).

**Solution :**
\[
x^4 + 27x = x(x^3 + 27) = x(x^3 + 3^3) = x(x + 3)(x^2 - 3x + 3^2) = x(x + 3)(x^2 - 3x + 9).
\]

**Example 16.** Resolve into factors : \(1 - 8a^3\).

**Solution :**
\[
1 - 8a^3 = 1^3 - (2a)^3 = (1 - 2a)\{1^2 + 1.2a + (2a)^2\} = (1 - 2a)(1 + 2a + 4a^2)
\]

**Exercise 3.3**

Resolve into factors :

1. \(3a^2b + 6ab^2 + 12a^2b^2\)
2. \(a(x + 5y) + 3b(x + 5y)\)
3. \(ax + by + bx + ay\)
4. \(1 + a + b + ab\)
5. \(ab + a - b + 1\)
6. \(a^2 + c^2 - 2ab + b^2\)
7. \((a^2 - b^2)(x^2 - y^2) + 4abxy\)
8. \((a + b + c)^3 - a + b^2 + 3\)
9. \(4x^2 + y^2 + z^2 + 2yz\)
10. \(a^4 + 4\)
11. \(x^4 + x^2 + 25\)
12. \(12a^4 + 3b^4\)
13. \(a^2 + b^2 + 2ac + 2bc\)
14. \(x^4 + 2x^2 + 9\)
15. \(a^4 + 27a^2 + 1\)
16. \(2ab + a^2 + b^2 + c^2\)
17. \(a^2 + b^2 + 2b\)
18. \((R + 2r^2 + t^2)\)
19. \(a^3 + 8\)
20. \(m^4 + 8m\)
21. \(x^3 + 3x^2 + 3x + 2\)
22. \(8 - a^3 + 3a^2b + 3ab^2 + b^3\)
23. \(a^3 + 9b^3 + (a + b)^3\)
24. \(m^3 + n^3 + m(n^2 + n^2) + n(m + n^2)\)
25. \(ay + a + \beta + 2y + \alpha + 1\)
26. \(\sqrt{2x + 2x^2}\)
27. \(x^3 + 3\)
28. \(AR^3 + AR^2h + AR^2h\)
29. \(x^2 + 3x + \alpha + \beta + \gamma + a + 2\)

[Hints : Given expression = \(x^2 + 2x + 2a + a + x + a + 2\).]
30. \( x(x + 3)(x + 4)(x - 1) + 4 \)
31. \( 16x^2 \cdot 25y^2 \cdot 8xz + 10yz \)
32. \( 4\pi(R + r)^3 \cdot 4\pi R^3 \)
33. \( \frac{1}{2}m(v + 2u)^2 \cdot \frac{1}{2}m(v + u)^2 \)
34. \( 2\sqrt{2}x^3 + 125 \)

**Factorisation of expressions of the form :** \( x^2 + px + q. \)

\[
x^2 + (a + b)x + ab = x^2 + ax + bx + ab
= x(x + a) + b(x + a)
= (x + a)(x + b)
\]

It is observed from here that

\( x^2 + px + q = (x + a)(x + b) \) if \( a \) and \( b \) be such that \( q = ab \) and \( p = a + b. \)

Hence, for resolving \( x^2 + px + q \) into factors, the term \( q \) independent of \( x \) is expressed as a product of two numbers \( a \) and \( b \) whose (algebraic) sum \( a + b \) is equal to coefficient of \( x \).

In this case,

(a) If \( q > 0, p > 0, \) \( a \) and \( b \) will be positive.

(b) If \( q > 0, p < 0, \) \( a \) and \( b \) will be negative.

(c) If \( q < 0, p > 0, \) the bigger one of \( a \) and \( b \) will be positive and the smaller one will be negative.

(d) If \( q < 0, p < 0, \) the bigger one of \( a \) and \( b \) will be negative and the smaller one will be positive.

It is to be noted that the coefficient of \( x^2 \) of quadratic expression under consideration is 1.

**Example 17.** Resolve into factors : \( x^2 \cdot D \cdot x \cdot D \cdot 12. \)

**Solution :** Here we are to find two numbers whose product is \( D12 \) and algebraic sum is \( D1. \) Such numbers are \( D4 \) and \( 3. \) Hence,

\[
x^2 \cdot D \cdot x \cdot D \cdot 12 = \frac{x}{D} \cdot D \cdot 4x + 3x \cdot D \cdot 12 = x(x \cdot D \cdot 4) + 3(x \cdot D \cdot 4) = (x \cdot D \cdot 4)(x + 3).
\]

**Explanation :** \( x^2 \cdot D \cdot x \cdot D \cdot 12 = \frac{x}{D} + (D1)x + (D12). \) Here, \( p = D1, q = D12. \)

**Example 18 :** Resolve into factors : \( x^4 + x^2 \cdot D \cdot 20. \)

**Solution :** \( x^4 + x^2 \cdot D \cdot 20 = x^4 + 5x^2 \cdot D \cdot 4x^2 \cdot D \cdot 20 \)

\[
= x^2(x^2 + 5) \cdot D \cdot 4(x^2 + 5) = (x^2 + 5)(x^2 \cdot D \cdot 4)
= (x^2 + 5)(x^2 \cdot D \cdot 2^2) = (x^2 + 5)(x + 2)(x \cdot D \cdot 2).
\]
Example 19. Resolve into factors: \((x^2 \cdot x^2 + 3(x^2 \cdot x)) \cdot 40\).

Solution: Let, \(x^2 \cdot x = a\)
\[\therefore \text{Given expression} = a^2 + 3a \cdot 40 = a^2 + 8a \cdot 5a \cdot 40\]
\[= a(a + 8) \cdot 5(a + 8) = (a + 8)(a \cdot 5)\]
\[= (x^2 \cdot x + 8)(x^2 \cdot x \cdot 5) \quad \text{[substituting the value of } a]\]

Example 20. Resolve into factors: \(x^2 \cdot x \cdot (a + 1)(a + 2)\).

Solution: Let, \(a + 1 = y\), then \(a + 2 = y + 1\)
\[\therefore \text{Given expression} = x^2 \cdot x \cdot y \cdot (y +1)\]
\[= x^2 \cdot x \cdot y^2 \cdot y = x^2 \cdot y \cdot y \cdot x \cdot y\]
\[= (x + y)(x \cdot y \cdot y) \cdot 1(x + y) = (x + y)(x \cdot y \cdot y \cdot 1)\]
\[= (x + a + 1)(x \cdot a \cdot 1 \cdot 1) \quad \text{[substituting the value of } y]\]
\[= (x + a + 1)(x \cdot a \cdot 2)\]

Alternative Method:
\[x^2 \cdot x \cdot (a + 1)(a + 2)\]
\[= x^2 \cdot (a + 2)x + (a + 1)x \cdot (a + 1)(a + 2)\]
\[= x(x \cdot a \cdot 2) + (a + 1)(x \cdot a \cdot 2)\]
\[= (x \cdot a \cdot 2)(x + a + 1)\]

Exercise 3.4

Resolve into factors:

1. \(x^2 + x \cdot 20\)
2. \(x^2 \cdot 8x \cdot 20\)
3. \(x^2 \cdot 12x + 20\)
4. \(x^2 \cdot 19x \cdot 20\)
5. \(x^2 \cdot 21x + 20\)
6. \(y^3 + 2y \cdot 3\)
7. \(u^2 \cdot 30u + 216\)
8. \(a^4 + 4a^2 \cdot 5\)
9. \(x^4 \cdot 10x^2 + 16\)
10. \(x^6 \cdot 7x^3 + 12\)
11. \(x^6y^6 \cdot x^3y^3 \cdot 6\)
12. \(a^8 \cdot a^4 \cdot 2\)
13. \((x + y)^2 \cdot 4(x + y) \cdot 12\)
14. \((x^2 + 2x)^2 + 12(x^2 + 2x) \cdot 45\)
15. \(y^2 \cdot 2ay + (a + b)(a \cdot b)\)
16. \(x^2 \cdot x \cdot (a + 5a + 6)\)
17. \(x^2 \cdot \left(a + \frac{1}{a}\right) \cdot k + 1\)
18. \(x^2 \cdot \left(\frac{2}{a} \cdot 3a\right) \cdot x \cdot 6\)
19. \(x^2 + x \cdot (a + 1)(a + 2)\)
20. \(x^4 + 3x^3 \cdot 5x^2 \cdot 15x\)
Factorisation of expressions of the form : \( px^2 + qx + r. \)

If \( px^2 + qx + r = (ax + b)(cx + d) = acx^2 + (bc + ad)x + bd, \) then, \( p = ac, \) \( q = bc + ad, \) \( r = bd. \)

Hence, \( p \times r = ac \times bd = bc \times ad. \)

It is to be observed that, \((ax + b)(cx + d)\) are the factors of \( px^2 + qx + r \) where \( pr = bc \times ad \) and \( q = bc + ad. \)

Therefore, to-determine factors of the expression \( px^2 + qx + r, \) two such factors of \( pr \) (the product of coefficient of \( x^2 \) and the term independent of \( x \)) are to be determined whose algebraic sum will be equal to \( q. \)

**Example 21.** Resolve into factors : \( 3x^2 + 7x + 4 \)

**Solution :**
\[
3x^2 + 7x + 4 = 3x^2 + 3x + 4x + 4 = 3x(x + 1) + 4(x + 1) = (x + 1)(3x + 4)
\]

**Example 22.** Resolve into factors : \( 3k^2 - 22k - 25 \)

**Solution :**
\[
3k^2 - 22k - 25 = 3k^2 + 3k - 25k - 25 = 3k(k + 1) - 25(k + 1) = (k + 1)(3k - 25)
\]

**Example 23.** Resolve into factors : \( x^2y^2 - xy - 72. \)

**Solution :**
\[
x^2y^2 - xy - 72 = x^2y^2 - 9xy + 8xy - 72 = xy(xy - 9) + 8(xy - 9) = (xy - 9)(xy + 8)
\]

**Example 24.** Resolve into factors : \( 4x^4 - 25x^2 + 36. \)

**Solution :**
\[
4x^4 - 25x^2 + 36 = 4x^4 - 16x^2 + 9x^2 + 36 = 4x^2(x^2 - 4) + 9(x^2 - 4) = (x^2 - 2^2)(4x^2 - 9) = (x^2 - 2^2)(2x + 3)(2x - 3)
\]

**Example 25.** Resolve into factors: \( 3(a^2 + 2a)^2 - 22(a^2 + 2a) + 40. \)

**Solution :**
\[
3(a^2 + 2a)^2 - 22(a^2 + 2a) + 40 = 3(x^2 - 22x + 40) = 3(x^2 - 22x + 100 - 60) = 3((x^2 - 22x + 100) - 60) = 3((x - 10)^2 - 60) = (3a^2 + 6a - 60)(a^2 + 2a - 10).
\]

[Supposing \( a^2 + 2a = x \)]
Example 26. Resolve into factors: $ax^2 + (a^2 + 1) x + a$.

Solution: 

$$ax^2 + (a^2 + 1)x + a = ax^2 + a^2x + x + a$$

$$= ax(x + a) + 1(x + a) = (x + a)(ax + 1).$$

Exercise 3.5

Resolve into factors:

1. $4a^2 + 11a + 6$
2. $7p^2 - 7p + 8$
3. $35x^2 + 12$
4. $5(x + y)^2 + 18(x^2 - y^2) - 8(x - y)^2$
5. $(a + b)x^2 - 2ax + (a - b)$
6. $(a - 1)x^2 + a^2xy + (a + 1)y^2$
7. $19x + 6 + 7x^3$
8. $6p^2 - 11p + 150$
9. $4(x + 1)(2x + 3)(3x + 2)(6x + 1)$
10. $(a - m)x^2 - (x - a)xy + (m - x)y$
11. $\frac{1}{2}p^2 - 3p + 4$
12. $3y^2 + 11y + 6$
13. $4x^2 + 5x + 6$
14. $a(a + 1)(a + 2)(a + 3)$
15. $(x + 1)(x + 3)(x - 4)(x - 6) + 24$

Remainder Theorem

Function: The concept of a function is fundamental in Mathematics. Let us clarify the concept by an example. Consider that, there are 40 students in your class and each of them brings 6 books. Can you say how many books will be brought next Saturday? The answer is no, because you cannot say how many students will be present on that day. If the number of students present is 30, then the number of books will be $30 \times 6 = 180$ and if the number of present is 23, then the number will be $23 \times 6 = 138$. The answer depends on the number of students present. If we take the number of students present to be $x$ then the number of books be $6x$. Here $x$ can be any integer from 0 to 40. Considering $y = 6x$, the value of $y$ for any such value of $x$ will be any number from 0 to 240. Here for each value of $x$, one and only one value of $y$ is obtained. In this case $y$ is a function of $x$ and the definition is expressed by writing $y = f(x)$ or $y = g(x)$, etc. $x$ is said to be the independent variable and $y$ the dependent variable. Consider another example. If $x$ is any number and $y$ is its square, then $y$ is a function of $x$ and we write $y = x^2$. The dependency of $y$ on $x$ is the main idea of
a function. Generally, the independent variable by $x$ and the related values of the function are denoted by $f(x)$, $g(x)$, $h(x)$ etc. Then $f(x)$, $g(x)$ etc. are said to be values of the function. For example, if $f(x) = 3x - 1$, $g(x) = x^2$, then we can find the related values of the functions for definite values of $x$. For example, if we put $x = 5$ in the above examples then we get $f(5) = 3(5) - 1 = 14$ and $g(5) = 5^2 = 25$.

**Polynomial** : If $a \neq 0$, then $ax + b$ is a simple (or of degree one) polynomial, $ax^2 + bx + c$ is a quadratic (or of degree two) polynomial; $ax^3 + bx^2 + cx + d$ is a polynomial of third degree (or of degree three). As the values of any polynomial depend upon $x$ we may consider it as a function of $x$. So a polynomial of any degree can be denoted by a function notation such as $f(x)$. $x$ is also called indeterminant. The remainder theorem gives the remainder when a polynomial $f(x)$ is divided by $(x - a)$ without performing actual division. The degree of the divisor polynomial $(x - a)$ is 1. If the divisor polynomial is a factor of the dividend polynomial then the remainder will be zero and if it is not so then the remainder will be a number other than zero. In both the cases, denoting the quotient by $h(x)$ and the remainder by $r$ we get, $f(x) = (x - a)h(x) + r$.

Substituting $x = a$ in both the sides we get $f(a) = (a - a)h(a) + r = 0$. Hence, $r = f(a)$. Thus if $f(x)$ is divided by $(x - a)$, then the remainder is $f(a)$. This formula is known as remainder theorem. Any polynomial $f(x)$ will be divisible by $(x - a)$ if and only if $f(a) = 0$. The formula is known as factorisation theorem.

**Corollary** : If $a \neq 0$, the polynomial $ax + b$ will be a factor of any polynomial $f(x)$ when $f\left(\frac{-b}{a}\right) = 0$

**Proof** : For, $ax + b = a\left(x + \frac{b}{a}\right)$ will be a factor of $f(x)$ if and only if $x + \frac{b}{a} = x - \frac{b}{a}$ is a factor of $f(x)$ i.e. if and only if $f\left(\frac{-b}{a}\right) = 0$.

**Use of the Remainder Theorem in Factorisation**

The expression $x - a$ will be a factor of a polynomial $f(x)$ iff $f(a) = 0$. Generally the expression $ax + b$ will be a factor of $f(x)$ if $f\left(\frac{-b}{a}\right) = 0$. Using this result, simple factors (if there be any) of polynomial of degree three or more can be determined the coefficients of polynomial are assumed to be all integers. If the coefficient of the highest degree of indeterminant in the polynomial is 1, then any simple factor of the polynomial will be of the form $x - a$ whereas is an integer
and factor of the constant term of the polynomial. If the coefficient at the highest
degree term is not the polynomial then any simple factor of 1, will be of the form
\(ax + b\) where \(a\) and \(b\) are integers, \(a\) is a factor of the coefficient of the highest
degree and \(b\) is a factor of the constant. It is to be noted that \(a\) and \(b\) may be
positive or negative integers. This method of determining the factors of
polynomial with the help of the remainder theorem is also called the vanishing
method.

**Example 27.** Resolve into factors : \(x^3 - x - 6\).

**Solution :** Here \(f(x) = x^3 - x - 6\) is a polynomial, the factors of the constant \(6\)
are \(\pm 1, \pm 2, \pm 3\) and \(\pm 6\)

Putting \(x = 1\) or \(x = -1\), we see that the value of \(f(x)\) is not zero.

Putting \(x = 2\), we get \(\tilde{f}(2) = 2^3 - 2 - 6 = 8 - 8 = 0\).

Hence, \(x - 2\) is a factor of \(\tilde{f}(x)\)

The other factors of \(\tilde{f}(x)\) can be determined in two ways;

- (i) Dividing \(\tilde{f}(x)\) by the determined factor,
- (ii) Rearranging and regrouping the terms of \(\tilde{f}(x)\) conveniently.

The second method is more attractive.

In the above example,

\[\tilde{f}(x) = x^3 - x - 6 = \tilde{x}(x - 2) + 2x(x - 2) + 3(x - 2)\]

\[= (x - 2)(x^2 + 2x + 3)\]

**N.B.** As \(x^2 + 2x + 3\) cannot be further resolved into factors, resolution into
factors of the given polynomial is complete.

**Example 28.** Resolve into factors : \(x^3 - 7xy^2 - 6y^3\)

**Solution :** Consider \(x\) as indeterminant and \(y\) as constant,

let \(\tilde{f}(x) = x^3 - 7xy^2 - 6y^3\)

then \(\tilde{f}(\text{Dy}) = \text{D} y^3 - 7(\text{D} y)y^2 + 6y^3\)

\[= \text{D} y^3 + 7y^3 - 6y^3 = 0\]

\(\therefore\) \(x \pm \text{D} (\text{Dy}) = x + y\) is a factor of \(\tilde{f}(x)\)

Now, \(x^3 - 7xy^2 - 6y^3 = x^2(x + y) - xy(x + y) - 6\tilde{y}(x + y)\)

\[= (x + y)(x^2 - xy - 6\tilde{y}) = (x + y)(x^2 - 3xy + 2xy - 6\tilde{y})\]

\[= (x + y)(x(\text{D} 3y) + 2y(x + 3y)) = (x + y)(x \text{D} 3y)(x + 2y)\]
Example 29. Resolve into factors: \(54x^4 + 27x^3a - 16x - 8a\)

Solution:

Let \(A(x) = 54x^4 + 27x^3a - 16x - 8a\),

then \(f\left(D_\frac{1}{2}a\right) = 54\left(D_\frac{1}{2}a\right)^4 + 27\left(D_\frac{1}{2}a\right)^3a - 16\left(D_\frac{1}{2}a\right)a - 8a = 0\)

\(\therefore\) \(x \left(D_\frac{1}{2}a\right) = x + \frac{1}{2}a\) i.e. \((2x + a)\) is a factor of \(f(x)\).

Now, \(54x^4 + 27x^3a - 16x - 8a = 27x^3(2x + a) - 8(2x + a) = (2x + a)(27x^3 - 8)\)

\(= (2x + a) \{(3x)^3 \div 2^3\} = (2x + a)(3x \div 2)(9x^2 + 6x + 4)\)

Exercise 3.6

Resolve into factors:

1. \(a^3 \div 21a \div 20\)
2. \(x^3 + 6x^2 + 11x + 6\)
3. \(a^3 \div 3a^2b + 2b^3\)
4. \(x^3 + 3x + 36\)
5. \(a^4 \div 4a + 3\)
6. \(2a^3 \div 3a^2 + 3a \div 1\)
7. \(x^3 \div 3x^2 + 4x \div 4\)
8. \(x^6 \div xs + x^4 \div x^3 + x^2 \div x\)
9. \(x^3 + 6x^2 + 11xy^2 + 6y^3\)
10. \(12 + 4x \div 3x^2 \div x^3\)
11. \(2x^4 \div 3x^3 \div 3x \div 2\)
12. \(3a^3 + 2a + 5\)

H.C.F. and L.C.M.

We are familiar with the method of finding the H.C.F. and the L.C.M. of different expressions. Here we briefly recapitulate the method.

**Method of finding H.C.F.** : The H.C.F. can be determined with the help of factors or by the method of division.

The method of finding H.C.F. with the help of factors is discussed here. The H.C.F., of the numerical coefficients of the given expressions is determined by the method used in Arithmetic. Then finding the probable common factors of the remaining parts, their H.C.F. is determined. Now the product of the H.C.F. of the coefficients and the H.C.F. of the remaining parts is the required H.C.F. of the given expressions.
Example 30. Find the H.C.F. of $3x^2y + 6xy^2$, $9x^4y^2 - 36x^2y^4$ and $9x^2y^2(x^2 + 6xy + 8y^2)$.

Solution:
1st expression: $3x^2y + 6xy^2 = 3xy(x + 2y)$
2nd expression: $9x^4y^2 - 36x^2y^4 = 9x^2y^2(x^2 - 4y^2) = 9x^2y^2(x + 2y)(x - 2y)$
3rd expression: $9x^2y^2(x^2 + 6xy + 8y^2)$
   $= 9x^2y^2(x^2 + 4xy + 2xy + 8y^2)$
   $= 9x^2y^2[(x(x + 4y) + 2y(x + 4y)]$
   $= 9x^2y^2(x + 4y)(x + 2y)$

Now the H.C.F. of (i) 3, 9 and 9 is 3.
(ii) the H.C.F of $xy$, $x^2y^2$ and $x^2y^2$ is $xy$;
The H.C. F. of (iii) $(x + 2y)$, $(x + 2y)(x - 2y)$ and $(x + 4y)(x + 2y)$ is $(x + 2y)$.
∴ Required H.C.F. = $3xy(x + 2y)$

Remark: In determining the H.C.F. the factors ±1 of any expression are not considered. For example, the H.C.F. of 6 and 8 is 2. Again the H.C.F. of $\pm 6$ and $\pm 8$ is also 2. This is also applicable in the case of L.C.M.

Example 31. Find the H.C.F. of $x^3 - x - 24$ and $x^3 - 6x^2 + 18x - 27$.

Solution: 1st expression = $x^3 - x - 24 = x^2(x - 3) + 3x(x - 3) + 8(x - 3)$
   $= (x - 3)(x^2 + 3x + 8)$
   [by applying remainder theorem]
2nd expression = $x^3 - 6x^2 + 18x - 27$
   $= x^2(x - 3) - 3x(x - 3) + 9(x - 3)$
   $= (x - 3)(x^2 - 3x + 9)$
   [by applying remainder theorem]
∴ Required H.C.F. = $(x - 3)$.

N.B. It is true that, the product of H.C.F. and L.C.M. of two algebraic expressions is equal to the product of the expressions (if ± sign is considered same in both cases). But for particular numerical values of letter symbols in the two algebraic expressions, the arithmetical H.C.F. (or L.C.M.) of the corresponding numbers may not be equal to the algebraic values of the H.C.F. (or L.C.M.) of the two expressions. For example, the H.C.F. of $(x + y)^2$, $x^2 - 1^2$ is $x + y$. But if $x = 6$ and $y = 4$, then the H.C.F. of the resulting numbers is 20 (which is double the corresponding numerical value of $x + y$).
Determination of L.C.M. : First the L.C.M. of numerical values of the coefficients of the given expressions is determined. Then the L.C.M. of remaining parts is determined by finding possible common factors of the remaining parts. Now the product of L.C.M. of coefficients and L.C.M. of the possible general factors of remaining parts is the required L.C.M.

Example 32. Determine the L.C.M. of \(2a^2b + 4ab^2\), \(4a^3b - 16ab^3\) and \(5a^3b^2(a^2 + 4ab + 4b^2)\)

Solution :
1st expression = \(2a^2b + 4ab^2 = 2ab(a + 2b)\);
2nd expression = \(4a^3b - 16ab^3 = 4ab(a^2 - 4b^2) = 4ab(a + 2b)(a - 2b)\);
3rd expression = \(5a^3b^2(a^2 + 4ab + 4b^2) = 5a^3b^2(a + 2b)^2\).
L.C.M. of 2, 4 and 5 is 20.
L.C.M. of \(ab(a + 2b), ab(a + 2b)(a - 2b)\) and \(a^3b^2(a + 2b)^2\)
\[= a^3b^2(a + 2b)^2 (a - 2b)\]
∴ Required L.C.M. = \(20a^3b^2(a + 2b)^2(a - 2b)\).

Exercise 3.7

Find the H.C.F. (Q. 1 to 4) :
1. \(x^2 + x, x^2 + 2x + 1\)
2. \(a - b, a^3 + b^3\)
3. \(a^2 - b^2 - c^2 - 2bc, b^2 - c^2 - a^2 - 2ca, c^2 - a^2 - b^2 - 2ab\).
4. \(x^2 - 11x + 30, x^3 - 4x^2 - 2x - 15\)

Find the L.C.M. (Q. 5 to 10)
5. \(x^2 + 3x + 2, x^2 \not{=} 1, x^2 + x \not{=} 2\)
6. \(x^3 \not{=} 1, x^3 + 1, x^4 + x^2 + 1\)
7. \(x^2 \not{=} x(a \not{=} c) \not{=} ac, \frac{3}{x} \not{=} x(a + c) + ac, ax^3 \not{=} a^3x\)
8. \(x^3 \not{=} x^2 \not{=} 3x \not{=} 9 \not{=} 2x^2 \not{=} 2x \not{=} 3\)
9. \(4x^2 + 8x \not{=} 12, 9x^2 \not{=} 9x \not{=} 54, 6x \not{=} 6x \not{=} 30x^2 + 24\)
10. \(x(4 \not{=} x^3), x^4 + 6x^3 + 8x^2, x^2 + 2x \not{=} 8\)
11. If \((x + a)\) is the H.C.F. of \(x^2 + px + q\) and \(x^2 + p\langle x + q\langle\) then prove that, \((p \not{=} p\langle)a = q \not{=} q\langle\).
Formation of algebraic formulas and their applications in solving real problems:
It is to be noted:
(a) If Tk. q is for person, then amount for n persons is \( A = Tk. qn \)
(b) If q is the amount of work done in every, then the amount of work done in d is \( W = qd \).
(c) If the velocity is q metre/hour, then the distance covered in t hours is \( D = qt \) metre.
(d) In q\% increase/dicrease, the increased/dicreased value of a is
\[
A = a \pm a \left( \frac{q}{100} \right) = a \left( 1 \pm \frac{q}{100} \right) (+ \text{ is applicable for increase and } - \text{ is applicable for decrease})
\]
(e) If Tk. r is the profit of unit capital in unit time, then at the end of time n the profit and increased capital of invested amount Tk. p will be I and A where
(i) In case of simple profit.
\[
I = Tk. Pnr.
A = P + I + Tk. P (1 + nr)
\]
(ii) In case of compound profit (when at the end of per unit time, profit is added with capital)
\[
A = Tk. P(1 + r)^n
\]
[It is to be noted, if profit is added with capital at the end of year,
Capital at the beginning \( P_0 = P \)
Capital at the end of first year \( P_1 = P_0 + P_0r = P_0(1 + r) = P(1 + r) \)
Capital at the end of second year \( P_2 = p_1 + p_1r = p_1(1 + r) = P(1 + r)^2 \)
Capital at the end of third year \( P_3 = P_2 + P_2r = P_2(1 + r) = P(1 + r)^3 \) and in this way Capital at the end of n th year \( A = P(1 + r)^n \)]
(f) If p litre of water flows in a rectangular water reservoir and q litre of water flows out in unit time, then in time t, the total pt litre of water flows in and qt litre of water flows out. Hence, if at the beginning \( Q_0 \) litre of water is in reservoir then at the end of time t the quantity of water is \( Q_t = Q_0 + pt - qt \) litre.

Example 33. If bus fare for per person is Tk. q, then for n persons what will be total fare?
A bus is hired for joining to picnic at Tk. 5,700/- on condition that every passengers will share the fare equally. The fare is increased at the rate of Tk.3 per head due to absence of 5 persons. How many persons went the bus?
**Solution**: If bus fare per person is Tk q, then the total fare for n persons is $A = Tk \cdot qn$.

Let the number of interested persons be $x$, then

<table>
<thead>
<tr>
<th>No. of passengers</th>
<th>Fare per person</th>
<th>Total fare</th>
</tr>
</thead>
<tbody>
<tr>
<td>Interested</td>
<td>$x$</td>
<td>$q$</td>
</tr>
<tr>
<td>Actual</td>
<td>$x - 5$</td>
<td>$q + 3$</td>
</tr>
</tbody>
</table>

By given conditions,

$qx = (q + 3)(x - 5) = 5700$

From $qx = (q + 3)(x - 5)$ we get,

$qx = qx - 5q + 3x - 15$

or, $5q = 3(x - 5)$

or, $q = \frac{3}{5}(x - 5)$

From, $qx = 5700$, we get, $\frac{3}{5}(x - 5) \cdot x = 5700$

or, $(x - 5)x = 5700 \times \frac{5}{3} = 9500$

or, $x^2 - 5x - 9500 = 0$

or, $(x - 100)(x + 95) = 0$

Since number of passengers is positive, hence $x + 95 \neq 0$

Therefore, $x - 100 = 0$ i.e. $x = 100$

∴ Actual number of passengers = $x - 5 = 100 - 5 = 95$

**Example 34.** Reza and Shuzon together can do a work in $x$ days. Shuzon can do the work alone in $y$ days. In how many days Reza can do the work alone?

**Solution**: Let Reza can do the work in $d$ days and daily amount of work alone = $r$

And daily amount of work done by Shuzon = $s$.

Then,

<table>
<thead>
<tr>
<th></th>
<th>Days of work</th>
<th>Total work</th>
</tr>
</thead>
<tbody>
<tr>
<td>Reza</td>
<td>$x$</td>
<td>$rx$</td>
</tr>
<tr>
<td>Shuzon</td>
<td>$x$</td>
<td>$sx$</td>
</tr>
<tr>
<td>Shuzon</td>
<td>$y$</td>
<td>$sy$</td>
</tr>
<tr>
<td>Reza</td>
<td>$d$</td>
<td>$rd$</td>
</tr>
</tbody>
</table>

By the given conditions,

$rx + sx = sy = rd = 1$
From, \(rx + sx = 1\) we get, \(r + s = \frac{1}{x}\)

From, \(sy = 1\) we get, \(s = \frac{1}{y}\)

\[\therefore r = \frac{1}{x} \cdot \frac{1}{y} = \frac{y \cdot \frac{1}{x}}{xy}\]

Then from, \(rd = 1\) we get, \(d = \frac{1}{r} = \frac{xy}{yd} \cdot \frac{1}{x}\)

\[\therefore \text{Reza can do the work in} \ \frac{xy}{yd} \text{ days.}\]

**Example 35.** A boat man can go \(x\) k.m against current in \(p\) hours. It takes \(q\) hours to cover the same distance in favour of current. Find the speed of current and boat.

**Solution :** Let, the speed of boat be \(b\) k.m / hour and current be \(c\) k.m / hour. Then the speed of boat in favour of current is \((b + c)\) k.m / hour. and the speed of boat against current is \((b - c)\) k.m/hour.

Since, distance traversed = speed \(\times\) time, hence

\[x = (b - c)p\]

\[x = (b + c)q\]

Then, \(b + c = \frac{x}{q} \ldots (i)\)

\[b - c = \frac{x}{p} \ldots (ii)\]

Adding, \((i)\) and \((ii)\) we get, \(2b = \frac{x}{q} + \frac{x}{p} = x\left(\frac{1}{q} + \frac{1}{p}\right)\)

or, \(b = \frac{x}{2}\left(\frac{1}{q} + \frac{1}{p}\right)\)

Subtracting \((ii)\) from \((i)\) we get,

\[2c = \frac{x}{q} - \frac{x}{p} = x\left(\frac{1}{q} - \frac{1}{p}\right)\]

or, \(c = \frac{x}{2}\left(\frac{1}{q} - \frac{1}{p}\right)\)

\[\therefore \text{Speed of current is} \ \frac{x}{2}\left(\frac{1}{q} - \frac{1}{p}\right) \text{ k.m/hour}\]

and speed of boat is \(\frac{x}{2}\left(\frac{1}{q} + \frac{1}{p}\right) \text{ k.m/hour}\).

**Example 36.** If number of telephone call be \(n\), cost of each call be Tk. \(p\), line rent be Tk. \(r\) and VAT be \(x\)%, then find the amount of VAT and telephone bill.
**Solution**: The charge given for line rent and call is Tk. \((r + np)\)

\[
\text{Amount of VAT} = \text{Tk.} \frac{r + np}{100}
\]

\[
\text{Amount of bill} = \text{Tk.} \left[ \left( r + np \right) + \left( r + np \right) \frac{x}{100} \right]
\]

\[
= \text{Tk.} \left( r + np \right) \left( 1 + \frac{x}{100} \right)
\]

**Example 37.** Salary of Matin is \(x\)% higher than that of Jalil. As a result Jalil's salary is \(y\)% less than that of Matin. Express \(y\) in terms of \(x\).

**Solution**: Let, Matin's salary be Tk. \(m\) and Jalil's salary be Tk. \(j\).

According to the problem,

\[
m = j + x\% \text{ of } j = j + \frac{jx}{100} = j \left( 1 + \frac{x}{100} \right)
\]

\[
j = m \ \text{y\% of } m = m \frac{my}{100} = m \left( 1 \frac{y}{100} \right)
\]

\[
\Rightarrow \quad m = m \left( 1 \frac{y}{100} \right) \left( 1 + \frac{x}{100} \right)
\]

\[
or, \quad 1 = \left( 1 \frac{y}{100} \right) \left( 1 + \frac{x}{100} \right)
\]

\[
or, \quad 1 \frac{y}{100} = \frac{1}{1 + \frac{x}{100}} = \frac{100}{100 + x}
\]

\[
or, \quad \frac{y}{100} = 1 \frac{100}{100 + x} = \frac{x}{100 + x}
\]

\[
\Rightarrow \quad y = \frac{100x}{100 + x}
\]

**Example 38.** If \(t\)% sale tax be paid on sale price and saller wants to make \(r\)% profit, then determine sale tax and sale price including tax of a commodity which costs Tk. \(a\).

**Solution**: At \(r\) % profit, sale price \(b = \text{cost price} + \frac{r}{100} \text{ of cost price.}

\[
= \text{Tk.} \left( a + a \times \frac{r}{100} \right) = \text{Tk.} \ a \left( 1 + \frac{r}{100} \right)
\]

Sale tax at \(t\)% be \(S = t \% \text{ of sale price}

\[
= \text{Tk.} \left( b \times \frac{t}{100} \right)
\]

\[
= \text{Tk.} \ a \left( 1 + \frac{r}{100} \right) \frac{t}{100}
\]
\[
\frac{at(100 + r)}{10000}
\]

\[= \text{Tk.} \frac{at(100 + r)}{10000}\]

\[\therefore \text{ Sale price with tax } = \text{ Sale price + Sale tax.}\]

\[= \text{Tk.} \left\{ b + b \times \frac{t}{100} \right\}
= \text{Tk.} \left\{ 1 + \frac{t}{100} \right\}
= \text{Tk.} a \left( 1 + \frac{r}{100} \right) \left( 1 + \frac{t}{100} \right)
= \text{Tk.} \frac{a(100 + r)(100 + t)}{10000}\]

**Example 39.** There are two pipes connected to a water reservoir. The reservoir is filled up at \(m\) minutes by the first pipe and becomes empty at \(n\) minutes by the second pipe. If the pipes are opened together, how long will it take to fill-up the empty reservoir? (here \(n > m\) is to be considered).

**Solution :** Let, \(p\) litre of water flows in per minute and \(q\) litre of water flows out per minute and let the capacity of reservoir be \(v\) litre.

Let, the empty reservoir be filled up at \(t\) minute if the pipes are opened at a time. Empty reservoir is filled up at \(m\) minutes by first pipe.

\[\therefore v = mp \hspace{1cm} \text{(i)}\]

Filled-up reservoir gets empty at \(n\) minutes by second pipe.

\[\therefore o = v Ð qn \hspace{1cm} \text{or, } v = qn \hspace{1cm} \text{(ii)}\]

Empty reservoir is filled up at \(t\) minutes by two pipes.

\[\therefore v = pt Ð qt \hspace{1cm} \text{or, } v = t(p Ð q) \hspace{1cm} \text{(iii)}\]

From (i), \(p = \frac{v}{m}\) and from (ii), \(q = \frac{v}{n}\)

From (iii), \(v = \left( \frac{v}{m} Ð \frac{v}{n} \right)t \hspace{1cm} \text{or, } l = \left( \frac{1}{m} Ð \frac{1}{n} \right)t = \frac{nDm}{mn}t\)

\[\therefore t = \frac{mn}{nDm} \text{ minutes.}\]

\[\therefore \text{ Required time } = \frac{mn}{nDm}\]

**Example 40.** Distance between two places A and B be \(d\) k.m. Mizan and Muzib started walking towards each other from A and B respectively at the same time and meet after \(t\) hours. After \(s\) hours of meeting, Mizan reached B. Determine their speeds.
**Solution**: Let speed of Mizan be \( u \) k.m./hour and speed of Muzib be \( v \) k.m./hour and they meet at \( c \). Then

<table>
<thead>
<tr>
<th></th>
<th>Speed</th>
<th>Time</th>
<th>Distance Traversed</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mizan</td>
<td>( u )</td>
<td>( t )</td>
<td>( AC = ut )</td>
</tr>
<tr>
<td>Muzib</td>
<td>( v )</td>
<td>( t )</td>
<td>( BC = vt )</td>
</tr>
<tr>
<td>Mizan</td>
<td>( u )</td>
<td>( s )</td>
<td>( CB = us )</td>
</tr>
</tbody>
</table>

According to the problem,
\[
ut + vt = d
\]
\[
ut + us = d
\]
i.e \((u + v)t = d\)...........................(i)
\[
u(t + s) = d \]...........................(ii)

From (ii), \( u = \frac{d}{t + s} \)

and from (i), \( u + v = \frac{d}{t} \)

\[
\therefore \; v = \frac{d}{t} \left(\frac{1}{t} - \frac{1}{t + s}\right) = \frac{ds}{t(t + s)}
\]

\[
\therefore \text{ Speed of Mizan } \frac{d}{t + s} \text{ km./hour and }
\]

speed of Muzib \( \frac{ds}{t(t + s)} \) km./hour.

**Example 41.** The cost of a boat be Tk. \( m \); express how much does it require to sell to make \( q\% \) profit by a formula. If \( m = 3600 \) and \( q = 40 \), determine sale price using the formula.

**Solution**: Let, sale price be Tk. \( S \).

Total profit = \( q\% \) of cost = Tk. \( m \times \frac{q}{100} \)

Now, sale price = cost + profit

Hence, \( S = m + \frac{mq}{100} = m \left(1 + \frac{q}{100}\right)\)

\[
\therefore \text{ Required formula, sale price } = \text{ Tk. } m \left(1 + \frac{q}{100}\right)
\]

If \( m = 3600 \) and \( q = 40 \), using the formula,

Sale price = Tk. \( 3600 \left(1 + \frac{40}{100}\right) = \text{ Tk. } \left(3600 \times \frac{140}{100}\right) = \text{ Tk. } 5040.\)
**Example 42.** If yearly rate of profit be 5 percent, that what would be profit of Tk. 750 in 4 years?

**Solution:** It is known, \( I = Pnr \) where \( r = \frac{s}{100} \)

Here, \( P = 750 \), \( n = 4 \), \( s = 5 \), \( \therefore r = \frac{5}{100} \)

\[ I = Pnr = 750 \times 4 \times \frac{5}{100} = 150 \]

**Answer:** Profit Tk. 150.

**Example 43.** How much money would be Tk. 1040 in 15 years at yearly rate of simple profit of Tk. 4 percent?

**Solution:** It is known, \( S = P(1 + nr) \)

Here, \( P(\text{Taka}) = \text{Capital} \), \( n(\text{year}) = 15 \), \( s(\text{Taka}) = 4 \)

\[ r(\text{Taka}) = \frac{4}{100} \]

Given, \( S(\text{Taka}) = 1040 \).

According to the problem,

\[ 1040 = P \left( 1 + 15 \times \frac{4}{100} \right) = P \times \frac{8}{5} \]

\[ \therefore P = \frac{1040 \times 5}{8} = 650 \]

**Answer:** Capital Tk. 650.

**Example 44.** Find the capital with profit and compound profit of Tk. 1000 in 2 years at yearly rate of compound profit of Tk. 5 percent.

**Solution:** It is known, \( C = P(1 + r)^n \) [where \( C \) is capital with profit at compound profit]

Given, \( P = 1000 \), \( r = \frac{5}{100} \), \( n = 2 \).

\[ \therefore C = 1000 \left( 1 + \frac{5}{100} \right)^2 = 1000 \times \frac{21}{20} \times \frac{21}{20} = 1102\frac{50}{20} \]

\[ \therefore \text{Capital with profit} = \text{Tk. 1102\frac{50}{20}} \]

\[ \therefore \text{Compound profit} = \text{Tk. (1102\frac{50}{20} \text{ } \Delta 1000)} \]

\[ = \text{Tk. 102\frac{50}{20}} \]
Exercise 3.8

1. What is the profit of Tk. 350 for 4 years at the rate of 3\%\,50\, percent per annum?

2. If the cost of an article be Tk. C, the rate of profit be r\%, then what will be sell price?

3. If a goat is sold at Tk. P, then the profit is x\%; what is the cost of the goat?

4. If the simple profit of Tk. x for 4 years is Tk. x at the rate of x\%, then find the value of x.

5. The population of a city is 70 lakh. If the rate of growth of the population in the city is 30 per thousand, then what will be the population of that city after 3 years? [Here the rule of compound profit is applicable].

6. What is the difference of simple and compound profits of Tk. 500 for 3 years if the rate of profit is 5\%?

7. If the difference of simple and compound profits of some principal for 2 years is Tk. 1 at the rate of 4\% profit, then what is the capital?

8. If the compounded amount is Tk. 650 at the end of one year and is Tk. 676 at the end of two years, then what is the capital?

9. Purchasing 2 oranges at Tk. 5, how many oranges are to be sold at Tk. 35 to make a profit of x\%?

10. If a goat is sold at a profit of 2x\%, then the price obtained is Tk. \frac{27x}{2} more than the price obtained by selling it at a loss of x\%, what is cost price of the goat?

11. The loss is r\% when lemons are sold n per taka. How many lemons per taka shall have to sell to make a profit of s\%?

12. The loss is x\% if lemons are sold 12 per taka. How many lemons are to be sold per taka to make a profit of 11x\%?

13. There are two pipes in a tank. If the first pipe is opened, the tank is filled up in 20 hours. The full tank becomes empty in 30 hours by the second pipe. If two pipes are opened at the same time, how much time will it require to fill up the empty tank?

14. There are three pipes attached to a drum. The drum is filled up by the first two pipes in p and q minutes respectively and the full drum becomes empty by the third pipe in r minutes. Three pipes are opened at the same time and after s minutes the third pipe is closed. How much time will it require to fill up the drum?

15. A can do a work in p days and B can do it in 2p days. They begin to do the work together but after a few days A leaves the work unfinished. The rest of the work is finished in r days by B. In how many days is the work finished?
16. Moti, Joti and Smirity together can finish a work in m days. Joti and Smirity together can do the work in n days. In how many days Moti alone can do the work?

17. The cost of a car is Tk. x. At what price should it be sold to make a profit of y%?

18. The salary of the brother is y% more than that of the sister; as a result, the salary of the sister is x% less than that of the brother. Express x as a function of y.

19. The distance between two places A and B is d k.m. Ashik and Rajib start at the same time towards each other from A and B respectively and they meet after t₁ hours. After t₂ hours of meeting, Ashik reaches the place B. Where are their speeds?

20. The value added tax (VAT) on sweets is x%. If a trader sells sweets of Tk. p, how much VAT he is to pay? If x = 15, p = 2300, then what is the amount of VAT?

21. If the number of calls from a telephone is 173, the charge per call is Tk. 1₇₀, the line rent is Tk. 140 and the VAT is 15%, then find the amounts of telephone bill and the VAT.

22. A bus was hired at the cost of Tk. 2400 and it was decided that every passenger would share the expense equally. 10 passenger did not join and such the fare increased by Tk. 8 per head. How many persons availed the bus? How much did everybody pay?

23. A boat man can row d k.m in t₁ hours against the current. To cover the same distance he takes t₂ hours in favour of the current. What are the speeds of the current and boat?

24. A benevolent society distributes p kg. rice in such a way that those who help in distribution get \( \frac{1}{8} \) part. The rest of the rice is distributed to m widows with children and n widows without any child. If each widow with children gets twice the quantities a widow without any child gets, then show that the quantities of rice a widow with children receives.

\[
\frac{p}{m} \left[ 1 \mathcal{D} \left( \frac{1}{8} + \left( 1 \mathcal{D} \frac{1}{8} \right) \frac{n}{2m + n} \right) \right] \text{ kg}
\]

If p = 112, m = 14 and n = 7, then what is the quantity of rice received by each widow with children?

[N.B. By considering mother in place of the helpers of the society, brothers in place of the widows with children and sister in place of the widows without any child, the shares of brothers and sisters according to Muslim laws can be determined by adapting the formula given in this problem.]
Multiple Choice Questions (MCQ) :

1. Which of the following indicates the value of \( \frac{1}{2} \{(a + b)^2 \mp (a \mp b)^2\} \)?
   A. 4ab  
   B. 2(a^2 + b^2)  
   C. 2ab  
   D. a^2 + b^2

2. Which one is the factorizing expression of \( m^4 + m^2 + 1 \)?
   A. \( (m^2 \mp m + 1) (m^2 + m \mp 1) \)  
   B. \( (m^2 + m \mp 1) (m^2 \mp m + 1) \)  
   C. \( (m^2 + m + 1) (m^2 + m + 1) \)  
   D. \( (m^2 \mp m + 1) (m^2 + m + 1) \)

Consider the following question:
\[ x + \frac{2}{x} = 3 \]

Based on the above equations, answer questions no. (3 Ñ 5) :

3. Which one of the following is the value of \( \left( x \mp \frac{2}{x} \right)^2 \)?
   A. 9  
   B. 5  
   C. 3  
   D. 1

4. Which of the following values of \( x \) satisfies the given equation?
   A. 1, 2  
   B. 2, 3  
   C. 1  
   D. 2

5. What is the value of \( x^3 + \frac{8}{x^3} \)?
   A. 1  
   B. 8  
   C. 9  
   D. 16

6. Given that :
   i. \( ab = \left( \frac{a + b}{2} \right)^2 \mp \left( \frac{a \mp b}{2} \right)^2 \)
   ii. \( a \mp 2 \) is a factor of \( a^2 \mp a + 6 \)
   iii. if Tk. \( x \) is the profit from unit capital in unit time and if B is the enhanced capital of Tk. \( y \) in time \( m \) then, \( B = y (1 + x)^m \)

Which of the above statement is correct ?
   A. i and ii  
   B. ii and iii  
   C. i and iii  
   D. i, ii and iii
7. What is the H.C.F of \(p(9 - p^2)\) and \(p^2 (p^2 + 6p + 9)\)?
   A. \(p^2 (p + 3)^2 (p - 3)\)
   B. \(p^2 (p^2 - 9) (p - 3)\)
   C. \(p(p + 3)^2 (p - 3)\)
   D. \(p(p + 3)\)

Creative Questions :

1. Srewoshi gets \(p\%\) more salary than Lira. Therefore Lira gets \(q\%\) less salary than Srewoshi. (Srewoshi's salary is Tk. \(S\) and Lira's salary is Tk. \(L\))
   A. Exhibit their salary as an algebraic expression.
   B. Express \(q\) as the function of \(p\) and \(p\) as the function of \(q\).
   C. If Lira's salary is Tk. 12000, \(p = 900\) and \(q = 50\), then how much is Srewoshi's salary?
   If \(p = x + 10\) and \(q = y + 20\), then what would be the revised function?

2. Given that : \(x = 2 + \sqrt{3}\)
   A. Find the value of \(\frac{1}{x}\) from above equations.
   B. Find the value of \(x^4 + \frac{1}{x^4}\)
   C. Show that, \(\left(x^2 \frac{1}{x^2}\right)\left(x^3 \frac{1}{x^3}\right) = 720\)

3. Given that : \(P(x) = x^3 + 6x^2 + 12x + 9\)
   \(Q(x) = 24 + 8x \div 6x^2 \div 2x^3\)
   \(R(x) = (a \div m) x^2 \div 3x (x \div a) + 9 (m \div x)\)
   A. Resolve \(P(x)\) in to factors.
   B. If \(Q(x) = 0\), then find the value of \(x\).
   C. Find the H. C. F. and L. C. M. of \(P(x)\), \(Q(x)\) and \(R(x)\).
Chapter IV

Exponents and Logarithm

Positive Integral Exponent

If n is positive integer greater than one, then \( a^n \) expresses the continued product of n factors, each of which is a. i.e. \( a^n \) is the continued products of a by n times. 

\[ a^n = a \times a \times a \times \ldots \times a \text{ (n times a)} \]

\( a^n \) is called n th power of a. But it is customary to say \( a^2 \) as the square of a and \( a^3 \) as the cube of a. In \( a^n \), n is called the exponent of a and a is the base. 

It is supposed that \( a^1 = a \). Defining \( a^n \) in this way for \( n = 1 \), the laws of exponent given below hold good for all positive integral values of m and n.

1. If a is any number and if m and n be any positive integers then 

\[ a^m \cdot a^n = a^{m+n} \]

Because, 

\[ a^m \cdot a^n = (a \times a \times a \times \ldots \times a) \cdot (a \times a \times \ldots \times a) \]

\[ = (a \times a \times a \times \ldots \times a) = a^{m+n} \]

Corollary: If \( m_1, m_2, \ldots, m_r \) are positive integers then 

\[ a^{m_1} \cdot a^{m_2} \ldots \ldots \ldots a^{m_r} = a^{m_1+m_2+\ldots+m_r} \]

2. If \( a \neq 0 \), m, n, \( (m \in \mathbb{N}) \) then

\[ \frac{a^m}{a^n} = \frac{a \times a \times \ldots \times a \text{ (m times a)}}{a \times a \times \ldots \times a \text{ (n times a)}} = a^{m-n} \]

3. If a, b be any numbers and if n be a positive integer then applying the commutative law for multiplication ab = ba we get,

\[ (ab)^n = (ab) \times (ab) \times \ldots \times (ab) \text{ (n times ab)} \]

\[ = a \times a \times \ldots \times a \times b \times b \times \ldots \times b = a^n \cdot b^n. \]
(4) If $a$ be any number, $b \neq 0$, and $n$ be any positive integer, \[
\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}, \quad \text{because}
\]
\[
\frac{a}{b} \times \frac{a}{b} \times \frac{a}{b} \times \ldots \ldots \times \frac{a}{b} = \frac{a \times a \times \ldots \times a}{b \times b \times \ldots \times b} = \frac{a^n}{b^n}
\]

**Negative Integral Exponent:**

**Definition of $a^{-1}$:** If $a \neq 0$, and $n$ be a negative integer, $a^{-n} = \frac{1}{a^n}$

Such as $a^{-1} = \frac{1}{a}$

$a^{-2} = \frac{1}{a^{-2}} = \frac{1}{a^2}$

It is to be noted, $n \in \mathbb{N}$, $a^{-n} = \frac{1}{a^n}$

Now the laws of exponents hold good for any positive or negative integral values of $m$ and $n$. For completeness, $a^0$ is to be defined where $a \neq 0$.

**Definition:** If $a \neq 0$, $a^0 = 1$.

**Laws of Exponents:** For any integers $m$, $n$,

\[
a^{m+n} = a^m a^n; \quad \frac{a^m}{a^n} = a^{m-n}; \quad (a^m)^n = a^{mn}; \quad a \neq 0
\]

\[
(ab)^n = a^n b^n; \quad \left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}, \quad b \neq 0
\]

**Example 1:** (i) $2^0 = 1$.  \(\) (ii) $2^4 = 2.2.2.2 = 16$  \(\) (iii) $3^{-3} = \frac{1}{3^3} = \frac{1}{27}$

**Example 2:**  \(\)

(i) $5^3 \times 5^5 = 5^{3+5} = 5^8 = (5^4)^2 = (625)^2 = 390625$

(ii) $5^3 \div 5^5 = 5^{3-5} = 5^{-2} = \frac{1}{5^2} = \frac{1}{25}$

(iii) $\left(\frac{3}{4}\right)^5 \times \left(\frac{3}{4}\right)^5 = \left(\frac{3}{4}\right)^{5+5} = \left(\frac{3}{4}\right)^0 = 1$

(iv) $6^3 = (2 \times 3)^3 = 2^3.3^3 = 216$

(v) $\left(\frac{2}{3}\right)^4 = \left(\frac{2^4}{3^4}\right) = \frac{16}{81}$

If $a$ is a positive real number and $n$ is a positive integer, then $n$th root of $a$ is a real number $x$ such that $x^n = a$. For every positive real number, there is a unique
positive nth root. It is denoted by $\sqrt[n]{a}$. Hence, $b = \sqrt[n]{a}$ means that $b > 0$ and $b^n = a$. If $a$ is a negative real number and $n$ is an odd natural number then there is a unique negative nth root of $a$, denoted by $\sqrt[n]{-a}$.

For example, $\sqrt[3]{27} = 3$, because $(3)^3 = 27$. If $a = 0$, then its nth root is also 0. That is $\sqrt[n]{a} = 0$.

If $n$ is positive or negative, proper or improper fraction (rational number), then we can now define $a^n$.

Let, $n = \frac{p}{q}$ where $p$, $q$ are integers and $q > 0$.

**Definition** : $a^{\frac{1}{q}} = \sqrt[n]{a}$, particularly $a^{\frac{p}{q}} = \sqrt[n]{a^p}$

For example, $8^{\frac{3}{4}} = (8^2)^{\frac{1}{4}} = (64)^{\frac{1}{4}} = 4$

The laws of exponent are also valid for rational exponents,

**Example 3** :  
(i) $8^{\frac{3}{4}} + 8^{\frac{1}{2}} = 8^\frac{3}{4} + \frac{1}{2} = 8^\frac{5}{4}$

(ii) $8^{\frac{3}{4}} \cdot 8^{\frac{1}{2}} = 8^{\frac{3}{4} + \frac{1}{2}} = 8^{\frac{5}{4}}$

(iii) $\left(10^{\frac{3}{4}}\right)^4 = 10^\frac{3}{4} \cdot 4 = 10^2$

(iv) $(50)^{\frac{1}{2}} = (25.2)^{\frac{1}{2}} = 25^{\frac{1}{2}} \cdot 2^{\frac{1}{2}} = 5 \cdot \sqrt{2}$

(v) $8^{\frac{5}{4}} = 8^{\frac{1+1}{4}} = 8^1 \cdot 8^{\frac{1}{4}} = 8 \cdot \sqrt[4]{8}$

**Example 4** : If $a$ be a positive real number and $l$, $m$, $n$ be rational numbers, then show that,

$$\left(\frac{a^m}{a^n}\right)\left(\frac{a^n}{a^l}\right)^m\left(\frac{a^l}{a^m}\right)^n = 1$$

**Solution** : Left hand side

$$= \left(\frac{a^m}{a^n}\right)\left(\frac{a^n}{a^l}\right)^m\left(\frac{a^l}{a^m}\right)^n$$

$$= a^{lmDn} \cdot a^{nmDl} \cdot a^{Dmn}$$

$$= a^{lmDn + mnDl + lnDmn} = a^0 = 1$$
Example 5: Simplify : \((12)^{\frac{1}{2}} \cdot \sqrt[3]{54}\)

Solution: \((12)^{\frac{1}{2}} \cdot \sqrt[3]{54} = \frac{1}{4} \cdot 2 \times 3^3 = \frac{1}{4} \times (2 \times 3)^3 = \frac{1}{2} \times 2^3 \times 3 = \frac{3^2}{2^3} = \frac{\sqrt{3}}{\sqrt{2^2}} = \frac{\sqrt{3}}{\sqrt{4}}\)

Exercise 4.1

Simplify (from 1 to 7)

1. \((a^{D1} + b^{D1})^{D1}\) \([a > 0; \ b > 0]\)

2. \((\frac{x^{D+q}}{x^{2q}}) \cdot (\frac{x^{q+r}}{x^{2p}}) \cdot (\frac{x^{r+p}}{x^{2q}})\) \([x > 0 \text{ and } p, q, r \text{ are rational numbers}]\)

3. \(\sqrt[3]{x^{D1}y} \cdot \sqrt[3]{y^{D1}z} \cdot \sqrt[3]{z^{D1}x}\) \([x > 0, \ y > 0, \ z > 0]\)

4. \((-\frac{x^a}{x^b})^{\frac{1}{ab}} \cdot (-\frac{x^b}{x^c})^{\frac{1}{bc}} \cdot (-\frac{x^c}{x^a})^{\frac{1}{ca}}\) \([x > 0 \text{ and } a, b, c > 0]\)

5. (i) \(\Pi_{4}^{3} \cdot \Pi_{4}^{3}\) (ii) \(\Pi_{4}^{3} \div \Pi_{4}^{3}\) (iii) \(\frac{4^n \cdot D_1}{2^n \cdot D_1}\)

6. \(\frac{3.2^n \cdot D_4.2^{nD2}}{2^n \cdot D_2^{nD1}}\)

7. \(\frac{2^{x+4} \cdot D_4.2^{x+1}}{2^{x+2} + 2}\)

8. \(\frac{2^{n+1} \cdot 3^{2nDm} \cdot 5^{m+n} \cdot 6^m}{6^n \cdot 10^{m+2} \cdot 15^n}\)

9. \(\frac{3^{m+1}}{(3m)^{nD1}} + \frac{9^{m+1}}{(3mD1)^{m+1}}\)

10. Show that, \(\left(\frac{x^q}{x^r}\right)^{q+rDp} \times \left(\frac{x^r}{x^p}\right)^{r+pDq} \times \left(\frac{x^p}{x^q}\right)^{p+qDr} = 1\)

11. Show that, \(\left(\frac{x^{(p+q)^2}}{x^{pq}}\right)^{pDq} \times \left(\frac{x^{(q+r)^2}}{x^{qr}}\right)^{qDr} \times \left(\frac{x^{(r+p)^2}}{x^{rp}}\right)^{rDp} = 1\)
Logarithms

Logarithms are widely used for finding the product and quotients of large numbers or values of powers of rational exponents. Let, \( a > 0, \ a \neq 1 \) and \( n \) is a positive number. If \( a^x = n \), then \( x \) is called the logarithm (briefly log) of \( n \) to the base \( a \) and it is written as \( x = \log_a n \). \( \log_a n \) is read as \( \log \) of \( n \) to the base \( a \). It is to be noted that, \( a^x = n \) and \( x = \log_a n \) are equivalent statements.

**Example 6.** \( \log_{10} 100 = \log_{10} 10^2 = 2 \), because \( 10^2 = 100 \)

\[ \log_3 \left( \frac{1}{9} \right) = 2, \text{ because } 3^2 = \frac{1}{9} \]

\[ \log_5 (5\sqrt{5}) = \frac{3}{2}, \text{ because } 5^{\frac{3}{2}} = 5.5^{\frac{1}{2}} = 5\sqrt{5} \]

\( x \) may be positive or negative, but \( a^x \) is always positive. That is why logarithms of only positive numbers are defined. There is no logarithms of zero or negative numbers.

**Example 7.** If \( \log_{2\sqrt{5}} 400 = x \), then find the value of \( x \).

**Solution :** From the definition of \( \log \), we get

\[
(2\sqrt{5})^x = 400 = 16 \times 25 = 2^4 \times 5^2 = 2^4 (\sqrt{5})^4 = (2\sqrt{5})^4
\]

\[ \therefore \ x = 4. \]

**Example 8.** If \( \log_x 324 = 4 \), then what is the value of \( x \)?

**Solution :** Since \( \log_x 324 = 4 \)

\[ x^4 = 324 = 3 \times 3 \times 3 \times 3 \times 2 \times 2 \]

\[ = 3^4 \times 2^2 = 3^4 (\sqrt{2})^4 = (3\sqrt{2})^4 \]

\[ \therefore \ x = 3\sqrt{2} \]

**N.B.** If \( a > 0, \ a \neq 1 \) and \( a^x = a^y \), then \( x = y \).

Again, if \( x \neq 0, \ a > 0, \ b > 0 \) and \( a^x = b^x \), then \( a = b \).

**Exercise 4.2**

1. Find the value of :
   
   (i) \( \log_2 16 \) \hspace{1cm} (ii) \( \log_6 6\sqrt{6} \) \hspace{1cm} (iii) \( \log_4 a^4 \) \hspace{1cm} (iv) \( \log_4 2 \)
   
   (v) \( \log_{12} \sqrt{12} \) \hspace{1cm} (vi) \( \log_5 3\sqrt{5} \) \hspace{1cm} (vii) \( \log_5 \left( \frac{3\sqrt{5}}{(\sqrt{5})} \right) \)
2. Find the value of $x$:
   (i) $\log_{10}x = 2$  
   (ii) $\log_{10}x = 2$  
   (iii) $\log_5 x = 3$  
   (iv) $\log_5 x = 2$  
   (v) $\log_x 25 = 2$  
   (vi) $\log_x \frac{1}{9} = 2$.

**Formulae of Logarithms**

Formulae of Logarithms are given here without proof. Here $a > 0$, $a \neq 1$.

**Formula 1.** If $M$ is positive and $r$ is any real number then, $\log_a M^r = r \log_a M$

**Formula 2.** $\log_a(MN) = \log_a M + \log_a N$

**Formula 3.** $\log_a \left(\frac{M}{N}\right) = \log_a M - \log_a N$

**Formula 4.** $\log_a M = \frac{\log_b M}{\log_b a}$

It is to be noted, $\log_a a = 1$ and $\log_a 1 = 0$ ($a > 0$, $a \neq 1$)

**N.B.** If base of log is not given, same base is to be considered throughout.

**Example 9.** Show that, $\log 21 = \log 7 + \log 3$.

**Solution :** $\log 21 = \log (7 \times 3) = \log 7 + \log 3$

**Example 10.** Show that, $5 \log 3 - \log 9 = \log 27$.

**Solution :** $5 \log 3 - \log 9 = \log 3^5 - \log 3^2 = \log 3^3 = \log 27$

**Example 11.** Simplify : $3 \log \frac{36}{25} + \log \left(\frac{2}{9}\right)^3 \neq 2 \log \frac{16}{125}$

**Solution :** $3 \log \frac{36}{25} + \log \left(\frac{2}{9}\right)^3 \neq 2 \log \frac{16}{125}$

\[
= \log \left(\frac{36}{25}\right)^3 + \log \left(\frac{2}{9}\right)^3 \neq 2 \log \left(\frac{16}{125}\right) \left(2^4\right) \\
= \log \left[\frac{36^3}{25^3} \times \left(\frac{2}{3^2}\right)^3 + \left(\frac{2^4}{3^3}\right)^2\right] \\
= \log \left[\left(\frac{2^6 \cdot 3^6}{5^6}\right)^3 \times \frac{2^3}{3^6} + \frac{2^8}{5^6}\right] \\
= \log \left(\frac{2^9 \cdot 3^6}{5^6} \times \frac{2^3}{3^6} \times \frac{5^6}{2^8}\right) \\
= \log \left(\frac{2^9}{2^8}\right) = \log (2^{9-8}) = \log 2^1 = \log 2
Exercise 4.3

Show that (Q. 1 Ð 5):

1. \( \log 12 = \log 3 + \log 4 \)
2. \( \log 360 = 3 \log 2 + 2 \log 3 + \log 5 \)
3. \( \log \frac{50}{147} = \log 2 + 2 \log 5 - \log 3 - 2 \log 7 \)
4. \( 3 \log 2 + \log 5 = \log 40 \)
5. \( 5 \log 5 - \log 25 = \log 125 \)
6. Simplify:
   (i) \( 7 \log \frac{10}{9} - 2 \log \frac{25}{24} + 3 \log \frac{81}{80} \)
   (ii) \( \log 5 + 16 \log \frac{16}{15} + 12 \log \frac{25}{24} + 7 \log \frac{81}{80} \)
   (iii) \( 7 \log \frac{16}{15} + 5 \log \frac{25}{24} + 3 \log \frac{81}{80} \)
   (iv) \( \frac{\log \sqrt{27} + \log 8}{\log 1.2} \)
   \( \log \sqrt{27} + \log 8 \)
   \( \log 1000 \)
   \( \log 1.2 \)
   (v) \( \log \frac{a^3b^3}{c^3} + \log \frac{b^3c^3}{d^3} + \log \frac{c^3d^3}{a^3} = 3 \log b^2c. \)

The Scientific or Standard Form of a Number:
The average distance of the earth from the sun is nearly 150000000 km. and the radius of a hydrogen atom is 0.0000000037 cm. There are uses of such large and small numbers in science. For convenience, these numbers are expressed in the form of \( a \times 10^n \), where \( 1 \leq a < 10 \) (i.e. a equal or greater than 1 but less than 10) and \( n \) is integer (positive, negative or zero). Such form of any positive number is known as its standard form or scientific form. For example, the standard form of 100000 is \( 10^5 \) and that of 0.00001 is \( 10^{-5} \). In both the cases \( a = 1 \) and hence it has been omitted. To express any negative number in standard form 'Ð' sign is to be placed before the standard form of its absolute value.

Example 12. The temperature of the centre of the sun is 15000000 degree centigrade. Express the temperature in scientific form.
Solution: \( 15,000,000 = 15 \times 1,000,000 = 15 \times 10^6 \)
\( = \frac{15}{10} \times 10 \times 10^6 = 1.5 \times 10^7 \)
Example 13. The distance of the sun from the Mercury is 58000000 km. Express this number in scientific form.

Solution: \(58000000 = \frac{58}{10} \times 10^6 = 5u8 \times 10^7\).

Example 14. Express 0.0000000037 in scientific form.

Solution: \(0.0000000037 = \frac{37}{10000000000} = \frac{37}{10^{10}} = 37 \times 10 \times 10^{D10} = 3u7 \times 10^{D9}\)

Example 15. Express in general decimal form (i) \(3u47 \times 10^6\) (ii) \(4u5 \times 10^{D6}\)

Solution: (i) \(3u47 \times 10^6 = 3u47 \times 1000000 = 347 \times 10000 = 3470000\).

(ii) \(4u5 \times 10^{D6} = 4u5 \times \frac{1}{10^6} = \frac{45}{10} \times \frac{1}{10^6} = \frac{45}{10^7} = \frac{45}{10000000} = 0u0000045\).

Exercise 4.4

Express in scientific form (Q. 1-8)
1. 735 2. 0.0176 3. 830 4. 0.0245
5. 0.0000051 6. 637,000,000,000
7. The distance of the sun from the Venus is 105,600,000 km.
8. The distance of the sun from the Neptune is 450,000,000 km.

Express in general decimal form (Q. 9-14)
9. \(10^3\) 10. \(10^{D6}\) 11. \(1u23 \times 10^4\) 12. \(9u873 \times 10^{D2}\)
13. \(1u32 \times 10^{D7}\) 14. \(3u356 \times 10^{D8}\)

Common Logarithms

In use, 10 is generally considered as the base of logarithms. Logarithms to the base 10 are called Common Logarithms. In this case base is normally omitted i.e. \(\log N\) is written for \(\log_{10}N\). If the scientific form of any positive number \(N\) is \(a \times 10^n\) then,

\[
\log N = \log (a \times 10^n) = \log a + \log 10^n = \log a + n = n + \log a
\]

It is to be noted that the common logarithm of any positive number \(N\) is expressed as the sum of two parts. One part is an integer (which is positive, negative or zero) and the other part is zero or a number lying between zero and
one. When expressed in this way, the integral part is called the characteristic of \( \log N \) and the other part is called the mantissa of \( \log N \).

If \( N = 10^n \), then the characteristic of \( \log N \) is \( n \) and its mantissa is zero.

**The Characteristic of Common Logarithms**

We know that, the characteristic of any number is the integral power of 10 in its standard form. Therefore, the characteristic of \( \log 2\text{ú}81 \) is 0, because \( 2\text{ú}81 = 2\text{ú}81 \times 10^0 \). The characteristic of \( \log 281 \) is 2, because \( 281 = 2\text{ú}81 \times 10^2 \). The characteristic of \( \log 0\text{ú}00281 \) is \( -3 \), because \( 0\text{ú}00281 = 2\text{ú}81 \times 10^{-3} \).

In the standard form of a positive number greater than 1, the integral power 10 is zero or a positive integer and it is 1 less than the number of digits to the left of the decimal in the given number. In the standard form of a number less than 1, the integral power of 10 is a negative integer and its absolute value is 1 greater than the number of zeros lying between the decimal point and first non zero digit to the right of decimal in the given number. Thus we have the following rules for finding the characteristics of common logarithms:

(a) The characteristic of the logarithm of any number greater than 1 is zero or a positive integer which is 1 less than the number of significant digits before decimal.

(b) The characteristic of the logarithm of any number less than 1 is a negative integer. Its absolute value is 1 greater than the number of zeros between decimal and first non zero digit to the right of decimal.

**Example 16.** Determine the characteristics of the log of the following numbers.

(i) 8350

(ii) 62ú37

(iii) 0ú000835

**Solution :**

(i) The number 8350 is greater than 1. There are four digits before decimal, since \( 8350 = 8350\text{ú}0 \). Hence the characteristics of \( \log 8350 \) is 4 Ð 1 = 3.

(ii) The number 62ú37 is greater than 1. There are two digits before decimal. Hence characteristics of \( \log 62ú37 \) is 2 Ð1 = 1.

(iii) The number 0ú000835 is smaller than 1. The first significant digit to right of decimal is 8 and there are three zeros between decimal and 8. Hence the absolute value of the characteristics of \( \log 0ú000835 \) is \( 3 + 1 = 4 \). Hence the characteristics of \( \log 0ú000835 \) is \( -4 \).
Log Table
The mantissa of the common logarithm of any number is a non-negative number less than 1. The mantissa is an irrational number and there is no easy method to find it.

The approximate value up to any decimal place of the mantissa of log of any number can be determined by using higher mathematics.

Common logarithms are used to perform big multiplication and division and to find the powers etc. with easy. In all these cases, the approximate values of the mantissa are required. As such the tables of the approximate values of mantissa has been prepared and such tables is called Log table. Usually the digits of mantissa are given in it and the decimal is to be supplied at the time of use. The Log table consisting of 5 digits are given at the end of this book i.e. the approximate values of mantissa are given upto 5 decimal places.

The numbers 10, 11, 12, ............. , 99 are written in the first column (extreme left) of the Log table. The next 10 columns to right of this column are the main Log table. At the top (from left to right), the digits 0, 1, 2,.........., 9 are written consecutively. To right of these 10 columns, there are 9 more columns separately, at the top of which 1, 2, 3, ............... , 9 are mentioned. These parts are called the mean difference table. The method of determining the mantissa from the Log table is explained here through an example.

Example 17. Determine the mantissa of 4857 from the Log table.
Solution : Along the row corresponding to 48 written in the extreme left column of the Log table, we find the number 68574 in the column of 5. It means that the mantissa of log of 4850 is 0.68574. To find the mantissa of log 4857 we consider the column 7 of mean difference in the right of the main table. We find 63 in the row of 48. It means when the number increases from 4850 to 4857, the mantissa of log increases by 0.00063.
Therefore, the mantissa of log 4857 = 0.68574 + 0.00063 = 0.68637.

Example 18. Determine log 0.000456.
Solution : The characteristic of log 0.000456 is 4.
The mantissa of log 0.000456 is 0.65896 from the Log table.
∴ log 0.000456 = 4 + 0.65896 = 4.65896.
Here the sign of the characteristic is written on the top of the characteristic, since 4ú65896 stands for 4 0ú65896.

If the logarithm of N is x i.e. if \( \log N = x \) then x is called antilog of N and we write \( N = \text{antilog } x \).

In using logarithms we are to find numbers whose logarithms are known, that is, we are to find antilogs. To solve such problems with ease, antilog table has been prepared like log table. If the mantissa of logarithm of any unknown number is known then we can find the number from the antilog table.

The first two digits of mantissa are given in the extreme left column of the antilog table, the 3rd digit is to be located from the top of the next ten columns and the 4th digit from the nine columns of the mean difference. It is to be noted that the decimal is also omitted here. The use of antilog is explained here through examples.

**Example 19.** The logarithm of a number is 0ú5514; find the number.

**Solution:** Let the number be \( x \), \( \therefore \log x = 0ú5514 \). The characteristic of \( \log x \) is 0 and mantissa is ú5514. The first two digits of the mantissa are 55. Let us consider the row along 55 placed in the extreme left column of antilog table. We find 35563 in the column of 1 in that row which means \( \log 3ú5563 = 0ú5510 \). Hence, \( \text{antilog } 0ú5510 = 3ú5563 \).

Then we find 33 in the column of 4 of the mean difference; it means if \( \log 0ú5510 \) increases to 0ú5514 then antilog increases to 0ú0033.

Hence, \( \text{antilog } 0ú5514 = 3ú5563 + 0ú0033 = 3ú5596 \),
\( \therefore x = 3ú5596 \).

**Example 20.** If \( \log x = 0ú5463 \), then find the value of \( x \).

**Solution:** The first two digits of mantissa is 54. Let us consider the row for 54 in the extreme left of the antilog table. We find 35156 in the column of 6 along that row. In the mean difference, we find 24 in the column of 3.
\( 35156 + 24 = 35180 \). The characteristic is 0ú3. Hence, there will be two zeros between decimal and the digits 35180.
\( \therefore x = 0ú003518 \).
Example 21. Find the value of $57{29}/1{904}$ upto two decimal places with the help of calculator and log tables.

Solution: By using calculator: $57{29}/1{904} = 109{08016} \approx 109{08}$.

By using log table:

$log (57{29}/1{904}) = log 57{29} + log 1{904}$

$= 1{75808} + 0{27964}$ (from the log table),

$= 2{03772}$.

Therefore, $57{29}/1{904} = antilog 2{03772} \approx 109{08}$ (from the antilog table).

Example 22. Find the capital with profit of Tk. 1000 of 2 years at the rate of 5% compound profit per annum.

Solution: We know, $C = P(1 + r)^n$.

Here $C$ is the amount (in taka) in the case of compound profit, $P = 1000$, $r = \frac{5}{100}$, $n = 2$.

$\therefore logC = logP(1 + r)^n = log P + n log(1 + r)$

$= log 1000 + 2 log 1{05}$

$= 3 + 2 \times 0{02119} = 3 + 0{04238} = 3{04238}$

From the log table, we get $antilog 3{04238} = 1102{50}$

$\therefore C < Tk. 1102{50}$

N.B. The result will be same if it is calculated by using calculators.

Example 23. Solve: $3^x = 16$.

Solution: $log 3^x = log 16$

or, $x log 3 = log 16$

or, $x = \frac{log 16}{log 3} < \frac{1{2041}}{0{4771}} < 2{52}$ [using calculator]

$\therefore x < 2{52}$
Exercise 4.5

(Calculators are to be used if the use of log table is not mentioned.)

1. Find the characteristics of logarithms of the following numbers:
   (i) 842  (ii) 75\(\times\)249  (iii) 7\(\times\)5249  (iv) 2\(\times\)329  (v) 0\(\times\)032  (vi) 0\(\times\)00021

2. Find the log of the following numbers (from the log table):
   (i) 324  (ii) 9\(\times\)27  (iii) 0\(\times\)04312.

3. Find the value of x from the following equations:
   (i) \(\log x = 0\(\times\)4871\)  (ii) \(\log x = 2\(\times\)54\)  (iii) \(\log x = \frac{2\sqrt{2}}{2}\) \(\times\) 6010

4. Find the product (approximate) in the following cases using log table:
   (i) 6\(\times\)79\(\times\)5\(\times\)34  (ii) 9\(\times\)56\(\times\)8\(\times\)72  (iii) 77\(\times\)5\(\times\)3\(\times\)7\(\times\)1\(\times\)4

5. Find the quotient (approximate) in the following cases using log table:
   (i) 3\(\times\)56\(\div\)2\(\times\)15  (ii) 293\(\div\)2\(\times\)212\(\times\)2

6. At the compound profit of 12%, what is the capital with profit of Tk. 273\(\times\)00 of 5 years?

7. In how many years any capital will be doubled at the rate of 5% compound profit?

8. The area of a rectangular field is 24 Ares. If the ratio of length and breadth is 3 \(\times\) 2, then what is the perimetre?

9. Solve : (i) \(4^{x+1} = 2^{3\times2}\)  (ii) \(3^{x} = 4^{2}\)

10. If \(\log 2 = 0\(\times\)3010\), \(\log 3 = 0\(\times\)4771\) and \(\log 7 = 0\(\times\)8450\) then find the values of the following expression using log table:
    (i) \(\log 6\)  (ii) \(\log 21\)  (iii) \(\log 42\).
Multiple Choice Questions (MCQ) :

1. Which one of the following is the correct value of \((a^{D}D)^{D1}\), if \(a \neq 0\)?
   A. \(a\)  
   B. \(a^{D1}\)  
   C. \(a^{D2}\)  
   D. \(a^{2}\)

2. Which one of the following is the correct value of \(\log_{4}64\)?
   A. 8  
   B. 4  
   C. 3  
   D. 2

3. Look at the following relations :
   i. \(\log \frac{A}{B} = \log A - \log B\)
   ii. If \(a^{z} = m\), then \(z = \log_{a} m\); where \(a > 0\), \(a \neq 1\) and \(m\) is positive integer.
   iii. If \(p\) and \(q\) are positive integers then \((a^{p})^{q} = a^{p+q}\); \(a \neq 0\)
   Which of the above statements is correct?
   A. i, ii and iii  
   B. i and ii  
   C. i and iii  
   D. ii and iii

4. Look at the following mathematical relations :
   i. There are logarithms of zero or negative numbers.
   ii. If \(y \neq 0\), \(a > 0\), \(b > 0\) and \(a^{y} = b^{y}\) then, \(a = b\)
   iii. If \(a > 0\), \(a \neq 1\) then, \(\log_{a} M^{q} = q \log_{a} M\)
   In the light of above mathematical relations, which one of the following is correct?
   A. i and ii  
   B. i and iii  
   C. ii and iii  
   D. i, ii and iii

Based on the following information, answer questions no. (5 Ñ 7)

\[ M = \frac{4^{m}D1}{2^{m}D1} \quad \text{and} \quad N = \frac{4^{m} + 1}{16^{m}}, \quad R = \log_{9}\sqrt{3} \]

5. Which one of the following is the correct value of \(M\)?
   A. \(2^{m} + 1\)  
   B. \(2^{m}D1\)  
   C. \(2^{m} + 1\)  
   D. \(2^{m}D1\)

6. Which one of the following is the correct value of \(\frac{M}{N}\)?
   A. \(2^{m}D1\)  
   B. \(2^{m} + 1\)  
   C. \(2^{m} + 1\)  
   D. \(2^{m}D1\)
7. Which one of the followings denotes the value of $M \times N \bar{R}$?
   A. $4.2^m + 1$
   B. $4(2^m \bar{1})$
   C. $4.2^m \bar{1}$
   D. $4(2^m + 1)$

Creative Questions:
1. If $p = x^a$, $q = x^b$ and $r = x^c$ then,
   A. Find the value of $\left(\frac{p}{q}\right)^c \times \left(\frac{q}{r}\right)^a \times \left(\frac{r}{p}\right)^b$
   B. Simplify: $2abc \left(\frac{p}{q}\right)^{\frac{1}{ab}} \times \left(\frac{q}{r}\right)^{\frac{1}{bc}} \times \left(\frac{r}{p}\right)^{\frac{1}{ca}} \times \sqrt{a^{D3}b^{D2}c} \times \sqrt{c^{D3}a}$
   C. Show that,
      \[
      \frac{(a \bar{D} b) \log(pq) + (b \bar{D} c) \log(qr) + (c \bar{D} a) \log(rp)}{\sqrt{a^{D1}b} \times \sqrt{b^{D1}c} \times \sqrt{c^{D1}a}} = 0
      \]
2. If $x = 2$, $y = 3$ and $z = 5$ then,
   A. Show that, $\log(x^3y^2z) = y \log x + x\log y + \log z$
   B. Simplify: $\log z + x^4 \log \frac{x^4}{yz} + x^2y \log \frac{z^2}{x^3y} + (x + z) \log \frac{y^4}{x^4z}$
   C. Find the value of $\log\sqrt{y^3} + y \log x \bar{D} \frac{y}{x} \log (xz)$
Chapter V
Ratio and Proportion

Ratio and Proportion are used in solving the problems of daily life. Consider the problem below:
Roni and Rana made a contract to complete a work for Tk. 160. Roni left the work after working for 6 hours alone. Rana finished the rest of the works in 10 hours. How much wages would each of them get?
If the wages be Tk. q per hour, then wages of Roni = Tk. 6q and wages of Rana = Tk. 10q.
\[ \therefore \quad 6q + 10q = 160 \]
or, \[ 16q = 160 \]
\[ \therefore \quad q = 10 \]
\[ \therefore \quad \text{Roni will get, Tk. } 6 \times 10 = \text{Tk. } 60 \]
and Rana will get, Tk. 10 \times 10 = Tk. 100
It is to be noted : \(60 = \frac{60}{100} \times 100 = \frac{3}{5} \times 100\). As a result Tk. 60 = \(\frac{3}{5}\) of Tk. 100;
Hence, wages of Roni \(\frac{3}{5}\) times of Rana's wages. We say, ratio of Roni's wages and Rana's wages is \(\frac{3}{5}\) and write, Roni's wages \(\frac{\text{Roni's wages}}{\text{Rana's wages}} = \frac{3}{5}\)

Ratio is used to compare two quantities of same kind in same unit. Ratio is always a number which may be integer or fraction (proper or improper). Ratio of two positive numbers \(a\) and \(b\) is \(\frac{a}{b} = \frac{a}{b}\)

Ratio of two expressions \(A\) and \(B\) of same kind is the ratio of their quantities in same unit.
If the quantity of \(A\) be \(a\) unit and quantity of \(B\) be \(b\) unit; \(\frac{A}{B} = a \frac{a}{b}\)

Ratio of two expressions is often written \(\frac{A}{B}\) to denote \(A \frac{a}{B}\). But if \(A\) and \(B\) be not numbers then \(\frac{A}{B}\) does not denote the operation of division.
In ratio \(A \frac{a}{B}\), \(A\) is called antecedent or first term and \(B\) is called consequent or second term.
If \(A \frac{a}{b}\), \(A \frac{ka}{kb}\), where \(k\) is a positive number.
Percentage is also a ratio whose consequent term is 100. Hence to transfer ratio into percentage, the consequent term is to transfer in 100.

Such as, \[ \frac{3}{5} = \frac{3 \times 20}{5 \times 20} = \frac{60}{100} = 60 \times \frac{1}{100} = 60\% \]

**Example 1.** If the perimeter of a square A be p units and that of a square B be r units (same unit), find the ratio of areas of the squares.

**Solution:** The perimeter of the square A = p units

∴ The length of a side of the square A = \( \frac{p}{4} \) units

∴ The area of the square A = \( \left( \frac{p}{4} \right)^2 \) or, \( \frac{p^2}{16} \) sq. units

The perimeter of the square B = r units

∴ The length of a side of the square B = \( \frac{r}{4} \) units

∴ The area of the square B = \( \left( \frac{r}{4} \right)^2 \) or, \( \frac{r^2}{16} \) sq. units

∴ The area enclosed by the square A \( \frac{\text{The area enclosed by the square B}}{\text{p}^2} = \frac{\text{r}^2}{16} = \frac{p^2}{16} \times \frac{r^2}{16} = \frac{p^2}{16} \times \frac{r^2}{16} \)

**Example 2:** A circle is inscribed in a square. What is the percentage of the area of the circle to the area of the square? [Find the result up to two decimal place.]

**Solution:** Let, one side of the square = 2r units.

∴ the diameter of the circle = 2r units

∴ the radius of the circle = \( \frac{2r}{2} \) units = r units

∴ The area of the circle \( \frac{\text{The area of the square}}{\pi r^2} = \frac{\pi}{4} = \left( \frac{3\sqrt{142}}{4} \times 100 \right)\% = (3\sqrt{142} \times 25)\% = 78.55\% \)

**Proportion**

If four quantities are such that the ratio of first and second quantities is equal to the ratio of third and fourth quantities then the four quantities are said to be in proportion. The four quantities of the proportion need not be of the same kind. The quantities of each ratio are to be of same kind.

a, b, c are said to be continued proportion if \( a \frac{\text{b}}{\text{b}} = b \frac{\text{c}}{\text{c}} \).

a, b, c are in continued proportion if and only if \( ac = b^2 \). In the continued proportion all the quantities are to be of the same kinds.
**Example 3.** A and B describe a particular distance with uniform speeds in times \( t_1 \) and \( t_2 \) minutes respectively. Find the ratio of the speeds of A and B.

**Solution:** Let the speeds of A and B be \( v_1 \) metre/minute and \( v_2 \) metre/minute.

\[ \therefore \text{ in } t_1 \text{ minutes A describes } v_1 t_1 \text{ metres.} \]

\[ \therefore \text{ in } t_2 \text{ minutes B describes } v_2 t_2 \text{ metres.} \]

According to the problem,

\[ v_1 t_1 = v_2 t_2 \]

\[ \therefore \frac{v_1}{v_2} = \frac{t_2}{t_1} \]

\[ \therefore \text{ Ratio of speeds } = \frac{t_2}{t_1} \]

**Transformation of Ratios**

Here ratio of expressions are positive numbers

1. If \( \hat{a} \hat{b} = c \hat{d} \), then \( b \hat{a} = d \hat{c} \) (invertendo)

**Proof:** Given that, \( \frac{a}{b} = \frac{c}{d} \). \( \therefore bc = ad \) [multiplying both sides by \( bd \)]

\[ \text{So, } \frac{bc}{ac} = \frac{ad}{ac} \] [none of \( a, b, c, d \) should be zero]

\[ \text{or, } \frac{b}{a} = \frac{d}{c} \text{ i.e. } b \hat{a} = d \hat{c} \]

2. If \( \hat{a} \hat{b} = c \hat{d} \), then \( a \hat{c} = b \hat{d} \) [alternendo]

**Proof:** Given that, \( \frac{a}{b} = \frac{c}{d} \). \( \therefore ad = bc \). \( \text{So, } \frac{ad}{cd} = \frac{bc}{cd} \)

\[ \text{or, } \frac{a}{c} = \frac{b}{d} \text{ i.e. } a \hat{c} = b \hat{d} \]

3. If \( \hat{a} \hat{b} = c \hat{d} \), then \( \frac{a + b}{b} = \frac{c + d}{d} \) [componendo]

**Proof:** Given that, \( \frac{a}{b} = \frac{c}{d} \). \( \therefore \frac{a}{b} + 1 = \frac{c}{d} + 1 \) [adding 1 to each side]

\[ \text{i.e. } \frac{a + b}{b} = \frac{c + d}{d} \]

4. If \( \hat{a} \hat{b} = c \hat{d} \), then \( \frac{a \hat{D} b}{b} = \frac{c \hat{D} d}{d} \) [dividendo]

**Proof:** Given that, \( \frac{a}{b} = \frac{c}{d} \). \( \therefore \frac{a}{b} \hat{D} 1 = \frac{c}{d} \hat{D} 1 \) [subtracting 1 from each side]

\[ \text{i.e. } \frac{a \hat{D} b}{b} = \frac{c \hat{D} d}{d} \]

5. If \( \hat{a} \hat{b} = c \hat{d} \), then \( \frac{a + b}{a \hat{D} b} = \frac{c + d}{c \hat{D} d} \) [componendo-dividendo]
Proof: If \( \frac{a}{b} = \frac{c}{d} \), then by dividendo we get, \( \frac{a + b}{b} = \frac{c + d}{d} \)

Hence, \( \frac{b}{a + b} = \frac{d}{c + d} \)

Again if \( \frac{a}{b} = \frac{c}{d} \), then by componendo we get, \( \frac{a + b}{b} = \frac{c + d}{d} \)

Hence, \( \frac{a + b}{b} \times \frac{b}{a + b} = \frac{c + d}{d} \times \frac{d}{c + d} \)

i.e. \( \frac{a + b}{a} = \frac{c + d}{c} \) [here \( a \neq b \) and \( c \neq d \) is to be considered]

6. If \( \frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \frac{g}{h} \), then each ratio \( \frac{a + c + e + g}{b + d + f + h} \)

Proof: Let, \( \frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \frac{g}{h} = k \)

\[ \therefore \quad a = bk, \ c = dk, \ e = fk, \ g = hk \]

\[ \therefore \quad \frac{a + c + e + g}{b + d + f + h} = \frac{bk + dk + fk + hk}{b + d + f + h} = \frac{k(b + d + f + h)}{b + d + f + h} = k \]

But \( k \) is equal to each of the ratio of the given proportion.

\[ \therefore \quad \frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \frac{g}{h} = \frac{a + c + e + g}{b + d + f + h} \]

Example 4. The sum of present ages of father and son is \( s \) years. \( t \) years ago, the ratio of their ages was \( r \) : \( p \). What will be the ratio of their ages after \( x \) year?

Solution: Let, present age of father be \( a \) years and that of son \( b \) years. Then, according to the problem, \( a + b = s \) .................(i)

\[ \frac{a + b}{b} \times \frac{b}{a + b} = \frac{r}{p} \] .................................(ii)

From (ii), \( \frac{a + b}{b} = \frac{r}{p} \)

\[ \therefore \quad \frac{a + b}{b} = \frac{s}{r + p} \]

\[ \therefore \quad a = \frac{(s + 2t)r}{r + p} + t \\
\text{and} \quad b = \frac{(s + 2t)p}{r + p} + t \]


\[
\begin{align*}
\therefore \text{ Ratio of ages of father and son after } x \text{ years} & = \frac{(s - 2t)r}{r + p} + t + x \\
& = \frac{(s - 2t)p}{r + p} + t + x
\end{align*}
\]

\[
\therefore \text{ Ratio of ages of father and son after } x \text{ year will be}
\]

\[
\left[ \frac{(s - 2t)r}{r + p} + t + x \right] \bigg/ \left[ \frac{(s - 2t)p}{r + p} + t + x \right].
\]

Example 5 : If \( \sqrt{5 + \sqrt{5}} \frac{x}{x} = 5 \), then what is the value of \( x \)?

Solution : Given, \( \sqrt{5 + \sqrt{5}} \frac{x}{x} = 5 \)

\[
\begin{align*}
\therefore & \frac{\sqrt{5 + \sqrt{5}} x}{5 \sqrt{5}} = 5 \\
& \frac{\sqrt{5 + \sqrt{5}} x}{5 \sqrt{5}} = \frac{5}{1} \text{ [by componendo and dividendo]} \\
& \text{or, } \frac{2\sqrt{5}}{2\sqrt{5}} = \frac{6}{4} = \frac{3}{2} \quad \text{or, } \frac{5}{5 \sqrt{5} x} = \frac{9}{4} \\
& \text{or, } 45 \sqrt{5} x = 20 \quad \text{or, } \sqrt{5} 9x = \sqrt{5} 25 \\
& \text{or, } x = \frac{25}{9} = 2\frac{7}{9} \quad \therefore x = \frac{7}{9}
\end{align*}
\]

Example 6 : Solve : \( \frac{a + x \sqrt{a^2 - x^2}}{a + x + \sqrt{a^2 - x^2}} = \frac{b}{x}, \quad 2a > b > 0 \)

Solution : Given that, \( \frac{a + x \sqrt{a^2 - x^2}}{a + x + \sqrt{a^2 - x^2}} = \frac{b}{x} \)

Hence, \( \frac{a + x \sqrt{a^2 - x^2}}{a + x + \sqrt{a^2 - x^2}} \frac{a + x + \sqrt{a^2 - x^2}}{a + x \sqrt{a^2 - x^2}} = \frac{b + x}{b \sqrt{a^2 - x^2}} \)

[by componendo and dividendo]

or, \( \frac{2(a + x)}{2\sqrt{a^2 - x^2}} = \frac{b + x}{b \sqrt{a^2 - x^2}} \quad \text{or, } \frac{(a + x)^2}{a^2 - x^2} = \frac{(b + x)^2}{(b \sqrt{a^2 - x^2})^2} \) [squaring both sides]

or, \( \frac{a + x}{a \sqrt{a^2 - x^2}} = \frac{b^2 + 2bx + x^2}{b^2 \sqrt{2bx + x^2}} \)

So, \( \frac{a + x + a \sqrt{a^2 - x^2}}{a + x \sqrt{a^2 - x^2}} = \frac{b^2 + 2bx + x^2}{b^2 + 2bx + x^2 \sqrt{2bx + x^2}} \)

[by componendo and dividendo]
or, \( \frac{2a}{2x} = \frac{2(b^2 + x^2)}{4bx} \) \quad or, \( \frac{a}{x} = \frac{b^2 + x^2}{2bx} \)

or, \( a = \frac{b^2 + x^2}{2b} \) \quad [multiplying both the side by x ]

or, \( x^2 + b^2 = 2ab \) \quad or, \( x^2 = 2ab - b^2 \)

\( \therefore \quad x = \pm \sqrt{2ab - b^2} \) \quad [here 2a > b is to be considered ]

**Example 7** : If \( \frac{x}{a} = \frac{y}{b} = \frac{z}{c} \), then prove that, \( \frac{x^3 + y^3 + z^3}{a^3 + b^3 + c^3} = \frac{xyz}{abc} \).

**Solution** : Let, \( \frac{x}{a} = \frac{y}{b} = \frac{z}{c} = k \quad \therefore \quad x = ak, \ y = bk, \ z = ck \)

L. H. S. = \( \frac{x^3 + y^3 + z^3}{a^3 + b^3 + c^3} = \frac{(ak)^3 + (bk)^3 + (ck)^3}{a^3 + b^3 + c^3} = \frac{a^3k^3 + b^3k^3 + c^3k^3}{a^3 + b^3 + c^3} = \frac{k^3(a^3 + b^3 + c^3)}{a^3 + b^3 + c^3} = k^3 \)

R. H. S. = \( \frac{xyz}{abc} = \frac{ak bk ck}{abc} = \frac{abck^3}{abc} = k^3 \)

\( \therefore \quad \text{L.H.S.} = \text{R.H.S.} \)

**Example 8** : If \( \frac{a + b}{b + c} = \frac{c + d}{d + a} \), then prove that, \( a = c \) or \( a + b + c + d = 0 \)

**Solution** : Given that, \( \frac{a + b}{b + c} = \frac{c + d}{d + a} \)

or, \( \frac{a + b}{b + c} \cdot 1 = \frac{c + d}{d + a} \cdot 1 \) \quad or, \( \frac{a + b}{b + c} \cdot \frac{d}{d + a} = 0 \)

or, \( \frac{a d c}{b + c} + \frac{a d c}{d + a} = 0 \) \quad or, \( (a d c)(d + a + b + c) = 0 \)

Either, \( a d c = 0 \) \quad i.e. \( a = c \)

or, \( a + b + c + d = 0 \)

**Example 9** : Using the properties of proportion, show that if

\( x = \frac{4ab}{a + b} \), then \( \frac{x + 2a}{x d a} + \frac{x + 2b}{x d b} = 2 \), \( a \neq b \).
Solution: Given that, \( x = \frac{4ab}{a + b} \) \[ \therefore \frac{x}{2a} = \frac{4ab}{2a(a + b)} \]

or, \( \frac{x}{2a} = \frac{2b}{a + b} \)

\[ \therefore \frac{x + 2a}{x} \bigg[ \frac{2b}{2a} \bigg] = 1 + \frac{b}{a} \] by componendo and dividendo

or, \( \frac{x + 2a}{x} \bigg[ \frac{3b + a}{b} \bigg] = 1 + \frac{3a}{b} \) by componendo and dividendo

Again, \( \frac{x}{2b} = \frac{2a}{a + b} \)

\[ \therefore \frac{x + 2b}{x} \bigg[ \frac{2a}{2b} \bigg] = 1 + \frac{a}{b} \] by componendo and dividendo

or, \( \frac{x + 2b}{x} \bigg[ \frac{3a + b}{a} \bigg] = 1 + \frac{3b}{a} \) by componendo and dividendo

Example 10: If \( ax = by = cz \), then show that,

\[ \frac{x^2}{yz} + \frac{y^2}{zx} + \frac{z^2}{xy} = \frac{bc}{a^2} + \frac{ca}{b^2} + \frac{ab}{c^2} \]

Solution: Let, \( ax = by = cz = k \) \[ \therefore x = \frac{k}{a}, \ y = \frac{k}{b}, \ z = \frac{k}{c} \]

L.H.S. \[ = \frac{x^2}{yz} + \frac{y^2}{zx} + \frac{z^2}{xy} = \frac{k^2}{a^2} + \frac{b}{k^2} \times \frac{c}{k^2} + \frac{k^2}{b^2} \times \frac{c}{k^2} + \frac{k^2}{c^2} \times \frac{ab}{k^2} \]

\[ = \frac{bc}{a^2} + \frac{ca}{b^2} + \frac{ab}{c^2} \]

= R.H.S.
Exercise 5.1

1. If the length of the sides of two squares be a metres and b metres, then what is the ratio of areas?

2. If the area enclosed by a circle is equal to the area enclosed by a square, then find the ratio of their perimeters.

3. The ratio of two natural numbers is $\frac{3}{4}$ and their L.C.M. is 180; find the two numbers.

4. If $x = 5$, then what is $3x$?

5. Express $3\frac{5}{4}$ in the form of $\frac{1}{x}$.

6. It is observed in a day that the ratio of the numbers of students absent and students present in your class is $\frac{1}{4}$. Express the number of students absent as the percentage of the total number of student.

7. A commodity is purchased and sold at a loss of 28%. Find the ratio of the sell price and the cost price.

8. The ratio of present ages of father and son is $\frac{7}{2}$ and after 5 years, the ratio of their ages will be $\frac{8}{3}$. What are their present ages?

9. A and B travels a particular distance with uniform speeds, in $t_1$ and $(t_1 + t_2)$ minute. Find the ratio of the speeds of A and B.

10. A pilliar of height $r$ metres stands at a distance of $p$ metres from a lamp post. If the height of the shadow of that pillar is $s$ metres, find out the height of the lamp. [Given that, the shadow is proportional to the height.]

11. If $a = b$, then establish the following:

   (i) $\frac{a}{c} = \frac{a^2 + b^2}{b^2 + c^2}$

   (ii) $\left(\frac{a + b}{b + c}\right)^2 = \frac{a^2 + b^2}{b^2 + c^2}$

   (iii) $a^2b^2c^2 \left(\frac{1}{a^3} + \frac{1}{b^3} + \frac{1}{c^3}\right) = a^3 + b^3 + c^3$

   (iv) $\frac{abc(a + b + c)^3}{(ab + bc + ca)^3} = 1$

   (v) $\frac{a \cdot b + c}{a} = \frac{(b \cdot c)^3}{c}$
12. Solve :
   (i) \( \frac{1 + \sqrt{1 + 1 \cdot x}}{1 + \sqrt{1 - x}} = \frac{1}{3} \)
   (ii) \( \frac{\sqrt{a + x} + \sqrt{a \cdot x}}{\sqrt{a + x} \cdot \sqrt{a \cdot x}} = b \)
   (iii) \( \frac{1 + \sqrt{1 + bx}}{1 + \sqrt{1 - bx}} = 1, \ 0 < b < 2a < 2b \)
   (iv) \( \frac{b + x + \sqrt{b^2 \cdot x^2}}{b + x \cdot \sqrt{b^2 \cdot x^2}} \)

13. If \( \frac{a}{b} = \frac{c}{d} \), then show that,
   (i) \( \frac{a^2 + ab + b^2}{a^2 \cdot ab + b^3} = \frac{c^2 + cd + d^2}{c^2 \cdot cd + d^3} \)
   (ii) \( \frac{a^2 + b^2}{a^2 \cdot b^2} = \frac{ac + bd}{ac \cdot bd} = \frac{c^2 + d^2}{c^2 \cdot d^2} \)

14. If \( \frac{a}{b} = \frac{b}{c} = \frac{c}{a} \), then prove that,
   (i) \( \frac{a^3 + b^3}{b^3 + c^3} = \frac{b^3 + c^3}{c^3 + d^3} \), (ii) \( (a^2 + b^2 + c^2) \cdot (b^2 + c^2 + d^2) = (ab + bc + cd)^2 \)

15. If \( \frac{\sqrt{1 + x} + \sqrt{1 \cdot x}}{\sqrt{1 + x} \cdot \sqrt{1 \cdot x}} = p \), then prove that, \( p^2 \cdot \frac{2p}{x} + 1 = 0 \).

16. If \( x = \frac{\sqrt{3m + 1} + \sqrt{3m \cdot 1}}{\sqrt{3m + 1} \cdot \sqrt{3m \cdot 1}} \), then prove that, \( x^3 \cdot 3mx^2 + 3x \cdot m = 0 \).

17. If \( x = \frac{\sqrt{2a + 3b} + \sqrt{2a \cdot 3b}}{\sqrt{2a + 3b} \cdot \sqrt{2a \cdot 3b}} \), then prove that, \( 3bx^2 \cdot 4ax + 3b = 0 \).

18. If \( \frac{a^2 + b^2}{b^2 + c^2} = \frac{(a + b)^2}{(b + c)^2} \), then prove that, \( a, b, c \) are in continued proportion.

19. If \( \frac{a^3 + b^3}{a \cdot b + c} = a(a + b) \), then prove that, \( a, b, c \) are in continued proportion.

20. If \( \frac{x}{b + c} = \frac{y}{c + a} = \frac{z}{a + b} \), then prove that, \( \frac{a}{y + z} \cdot \frac{1}{x} = \frac{b}{z + x} \cdot \frac{1}{y} = \frac{c}{x + y} \cdot \frac{1}{z} \).
21. If \( \frac{bz}{a} = \frac{cy}{b} = \frac{ax}{c} \), then prove that, \( \frac{x}{a} = \frac{y}{b} = \frac{z}{c} \).

22. If \( \frac{a + b}{a + b} = \frac{b + c}{b + c} = \frac{c + a}{c + a} \) and \( a + b + c \neq 0 \), then prove that, \( a = b = c \).

23. If \( \frac{x}{y} = \frac{a + 2}{a} \), then what is the value of \( \frac{x^2}{x + y^2} \)?

24. If \( \frac{x}{xa + yb + zc} = \frac{y}{ya + zb + xc} = \frac{z}{za + xb + yc} \) and \( x + y + z \neq 0 \), then show that the value of each ratio is \( \frac{1}{a + b + c} \).

25. If \( (a + b + c)p = (b + c - a)q = (c + a - b)r = (a + b - c)s \), then prove that,
\[
\frac{1}{q} + \frac{1}{r} + \frac{1}{s} = \frac{1}{p}
\]

26. If \( \frac{x}{y + z} = \frac{y}{z + x} = \frac{z}{x + y} \) and \( x, y, z \) be not all equal to each other, then prove that the value of each ratio will be equal to \( 1 \) or \( \frac{1}{2} \).

[Hints: Let \( \frac{x}{y + z} = \frac{y}{z + x} = \frac{z}{x + y} = k \) and \( x \neq y \), then \( x = k(y + z) \), \( y = k(z + x) \) and hence \( x \neq y \).]

27. If \( lx = my = nz \), then show that, \( \frac{x^2}{yz} + \frac{y^2}{zx} + \frac{z^2}{xy} = \frac{mn}{l^2} + \frac{nl}{m^2} + \frac{lm}{n^2} \).

28. If \( ax = by = cz \), then show that, \( \frac{yz}{x^2} + \frac{zx}{y^2} + \frac{xy}{z^2} = \frac{a^2}{bc} + \frac{b^2}{ca} + \frac{c^2}{ab} \).

29. Solve: 
   (i) \( \sqrt{\frac{x}{1}} + \sqrt{\frac{x}{6}} = 5 \)
   (ii) \( \sqrt{\frac{ax+b}{\sqrt{ax+b}}} + \sqrt{\frac{ax+b}{\sqrt{ax+b}}} = c \)
   (iii) \( 81\left(\frac{1}{1+x}\right)^3 = \frac{1+x}{1+x} \)
Successive Ratio

Suppose, the income of A is Tk. 1000, the income of B is Tk. 1500 and the income of C is Tk. 1125.

Income of A \(= \frac{1000}{1500} = \frac{2}{3}\)
Income of B \(= \frac{1500}{1125} = \frac{4}{3}\)

\[\therefore \text{income of A} : \text{income of B} : \text{income of C} = 8 : 12 : 9\]

If the two ratios are \(A : B\) and \(B : C\), then they are usually written as \(A : B : C\). This is called a successive ratio. Any two (more) given ratios can be expressed in this way. It may be noted that to express two ratios as \(A : B : C\) the consequent of the first ratio must be equal to the antecedent of the second ratio. For example, to express two ratios \(2 : 3\) and \(4 : 3\) as \(A : B : C\), the consequent of the first ratio will have to be made equal to the antecedent of the second ratio.

Now, \(2 : 3 = \frac{2 \times 4}{3 \times 4} = \frac{8}{12}\); Again \(4 : 3 = \frac{4 \times 3}{3 \times 3} = \frac{12}{9}\)

So, two ratios \(2 : 3\) and \(4 : 3\) in the form \(A : B : C\) will be \(8 : 12 : 9\).

Note that, \(A : C = \frac{1000}{1125} = \frac{8}{9}\), which is equal to the ratio derived from the form \(A : B : C = 8 : 12 : 9\).

Example 11. If A, B and C be of same type of quantities and \(A : B = 3 : 4\), \(B : C = 5 : 6\), then what is \(A : B : C\)?

Solution : \(\frac{A}{B} = \frac{3}{4} = \frac{3 \times 5}{4 \times 5} = \frac{15}{20}\), Again \(\frac{B}{C} = \frac{5}{6} = \frac{5 \times 4}{6 \times 4} = \frac{20}{24}\)

\[\therefore A : B : C = 15 : 20 : 24.\]

Example 12. If the ratio of three angles be \(3 : 4 : 5\); express three angles in degrees.

Solution : Sum of three angles = 180¡

Let, three angles according to given ratios be 3x, 4x and 5x.

By the problem, \(3x + 4x + 5x = 180\)

or, \(12x = 180\)

or, \(x = 15\)

Therefore, three angles are \(3x = 3 \times 15 = 45\)
\(4x = 4 \times 15 = 60\)
\(5x = 5 \times 15 = 75\)
Proportional division

To divide a quantity in a definite ratio is called proportional division. To divide $s$ according to $a : b : c : d$, $s$ is to be divided into $(a + b + c + d)$ parts and $a$, $b$, $c$ and $d$ parts are to be taken respectively. So the required

1st part $= \frac{a}{a + b + c + d}$ of $s = \frac{sa}{a + b + c + d}$

2nd part $= \frac{b}{a + b + c + d}$ of $s = \frac{sb}{a + b + c + d}$

3rd part $= \frac{c}{a + b + c + d}$ of $s = \frac{sc}{a + b + c + d}$

4th part $= \frac{d}{a + b + c + d}$ of $s = \frac{sd}{a + b + c + d}$

In this way any quantity can be divided in any number of definite ratios.

Example 13. Divide Tk. 5100 among three person so that,

share of 1st person $: share$ of 2nd person $: share$ of 3rd person $= 1 : 2 : 3$

Solution: $\frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{9} = \left(\frac{1}{2} \times 18\right) \left(\frac{1}{3} \times 18\right) \left(\frac{1}{9} \times 18\right) = 9 \cdot 6 \cdot 2$

Suppose, amount of total Taka = $S$ and the ratio of Taka received by three person = $a : b : c$.

.: Share of 1st person $= \frac{a}{a + b + c}$ of $s = \frac{9}{9 + 6 + 2}$ of 5100 $= \frac{9}{17}$ of 5100 $= \text{Tk. 2700}$

Share of 2nd person $= \frac{b}{a + b + c}$ of $s = \frac{6}{9 + 6 + 2}$ of 5100 $= \frac{6}{17}$ of 5100 $= \text{Tk. 1800}$

Share of 3rd person $= \frac{c}{a + b + c}$ of $s = \frac{2}{9 + 6 + 2}$ of 5100 $= \frac{2}{17}$ of 5100 $= \text{Tk. 600}$

Answer: Tk. 2700, Tk. 1800, Tk. 600.
Exercise 5.2

1. Divide Tk. 860 among Aziz, Abed and Ashique in such a way that, if Aziz gets Tk. 5, then Abed gets Tk. 4 and if Abed gets Tk. 3, then Ashique gets Tk. 4.

2. Divide Tk. 300 among A, B, C and D in such a way that the share of A is 2/3 times the share of B, the share of B is 1/2 times the share of C and the share of C is 3/2 times the share of D.

3. Three fishermen caught 690 fishes. If the ratio of their shares be, 2/3, 4/5 and 5/6, how many fishes did each of the fishermen catch?

4. If total score of Bulbul, Nannu and Akram in cricket game be 171 runs. If the ratio of runs of Bulbul, Nannu and Nannu, Akram be 3/2, what is their individual runs?

5. There are 2 officers, 7 clerks and 3 peons in a office. If a peon gets Tk. 1, a clerk gets Tk. 2 and an officer gets Tk. 4. If their total salary is Tk. 50000, how much salary does each of them get?

6. Rayhana Begum left Tk. 24075 at the time of her death, The cost of funeral ceremony was Tk. 675. The rest of the money was divided among husband, mother and two daughters in the ratio of 1/4, 1/6 and 1/3. How much did each of the daughter get?

7. In electing the leader of an association. Mr. Saem won by 4/3 votes. If the total number of the members be 581 and 91 members absent from voting, then by how much margin of votes the opponent of Mr. Saem was defeated?

8. If the cost price is 5/6, then what is the percentage of profit?

9. If the previous price of paper is 2/3, then what is the percentage of price increased compared to the previous price?

10. If the length of each side of a square is increased by 10%, then what is the percentage of increase in the area enclosed by the square?
11. The estimated cost of constructing a wooden bridge is Tk. 90,000. But Tk. 21,600 was spent more than the estimated cost. What is the percentage of increase in the cost?

12. The ratio of rice and husk be $\frac{7}{3}$, what will be the percentage of rice?

13. The ratio of production before and after the introduction of irrigation facilities in the cultivation of a field be $\frac{4}{7}$. Before introduction of irrigation the amount of paddy cultivated in a plot in the field was 30.4 quintal. How much will be the production of paddy after the introduction of irrigation?

14. The ratio of quantity of paddy and the quantity of rice produced from that quantity of paddy is $\frac{3}{2}$ and the ratio of a quantity of wheat and the quantity of flour produced from the quantity of wheat is $\frac{4}{3}$. Find the ratio of the quantity of rice produced from 1 quintal of paddy and the quantity of flour produced from 1 quintal of wheat.

15. The weight of 1 cubic cm. of a piece of wood is 7 decigram. What percentage is the weight of the piece of wood in comparison with the weight of equal volume of water?

16. The area of a plot of land is 588 sq. metre. The ratio of length and breadth of a second plot of land with that of the former plot of land is $\frac{3}{4}$ and $\frac{2}{3}$ respectively. Find the area of the latter plot of land.

17. Mr. Reza and Mr. Manzu took loan of different amounts at 10% simple profit from the same bank on the same day. The amount paid back for principal with profit by Reza after 2 years is the same by Manzu after 3 years. What was the ratio of paid their loans?

18. The perimeter of a triangle is 18 cm. If the ratio of length of the sides is $\frac{3}{4}$, then find the length of each side.

19. Divide Tk. 674 in the ratio $\frac{3}{4}$, $\frac{4}{5}$, $\frac{6}{7}$

20. If the ratio of two numbers be $\frac{5}{6}$ and their H.C.F be 4, what is the L.C.M of these two numbers?
Multiple Choice Questions (MCQ) :
1. The inverse ratio of $x \div y$ isÑ
   A. $x \div y$  
   B. $y \div x$  
   C. $\frac{1}{x} \div \frac{1}{y}$  
   D. $\sqrt{x} \div \sqrt{y}$

2. Given that :
   i. If $a \div b = b \div c$ then, $ac = b^2$
   ii. If $\frac{x}{y} = \frac{p}{q}$ then, $\frac{x + y}{x} = \frac{p + q}{q}$
   iii. If $m \div n = x \div y$ then, $mx = ny$

   Based on the above statements, which of the following is correct? 
   A. i and iii  
   B. i and ii  
   C. ii and iii  
   D. i, ii and iii

Answer questions no. (3 Ñ 5) based on the following information :
A circle is incscribed in a square. Radius of the circle is $r$.

3. Which one of the followings denotes the perimeter of the circle?
   A. $4\pi r^2$  
   B. $\pi r^2$  
   C. $2\pi r$  
   D. $2\pi r^2$

4. Which one of the followings is the ratio of the area of circle and square?
   A. $4 \div \pi$  
   B. $\pi \div 4$  
   C. $2 \div \pi$  
   D. $r \div 2$

5. Which one of the following denotes the length of a diagonal of the square?
   A. $2r$  
   B. $2 \sqrt{2}r$  
   C. $4r$  
   D. $4 \sqrt{2}r$

Creative Questions :
1. The ratio of length and diagonal of a rectangular land is $\frac{1}{5} \div \frac{1}{4}$
   A. Draw the land with diagonals and express the given ratio in the form of a $\div$.
   B. Determine the ratio of length, width and diagonal of the land.
   C. If the area of a rectangular land is 192 sq. metre, then find the area of a square whose perimeter is as same as the rectangle.
Chapter VI
Mathematical Open Sentences related to Single Variable

Mathematical sentences are constructed using different word or words and verbs like word or words and verbs are used for construction of sentences.

In mathematical sentences, different symbols are also used. Such as letter symbols N, Z, Q, R etc. are used to denote sets, numbers are used to denote expressions and operation like 5 + 8, 2 × 3 etc. When such mathematical words are joined be verbs, they form mathematical sentences. The verbs of mathematics are "equal to", "greater than", "smaller than" etc. or their symbols. For example, 5 + 8 = 13, 2 × 3 > 4, 10 < 13 are mathematical sentences.

Set is discussed earlier. If we write, A = \{ x \in R : 1 \leq x \leq 20 \}, then x \in R means that the value of x is any real number from 1 to 20. The domain of x consists of all numbers from 1 to 20. In this case, x is called a variable. A symbol which can assume any value from a particular set is called a variable. The set from which the variable takes its values is called the domain of the variable.

It is to be noted that whether the sentences x + 3 = 10 is true or false cannot be determined correctly without knowing the value of x. In this sentences, x is unknown and it represents a number. There is a domain for x from where x takes its values. Generally, R (the set of real numbers) is considered as the domain of x. But in some cases, Q (the set of rational numbers) is used as domain of x. In the above sentence, x is a variable and its domain is R. The above sentence is true if x takes the value 7 from R.

A mathematical sentence with some variables is called an open sentence. A mathematical sentence is called a mathematical statement if it is possible to say with certainly whether it is true or false. For example, 2 + 3 = 5, 8 ÷ 3 = 5 are mathematical statement : x + 12 = 17 is a mathematical open sentence.

Open sentences involving the sign equality are called equations. The values of the variables in an open sentence for which it becomes true are called the roots of the equation. The set of roots of the equation is called its solution set.

For example, x + 3 = 10 is an equation. The solution set of this equation is \{ 7 \}, because the sentence x + 3 = 10 is true only for x = 7. The sentence x + 3 = 10 may represent various types of problems. For example,
"How much is to be added to 3 to give 10?"

"Musa has Tk. 3, how much to be added to make Tk. 10?"

"Shima has 3 blouses; how many more is required to make 10 blouses?"

"There are three passengers in a Tempo, how many more passengers will make
10 passengers?" etc.

The expression on the left of the equality sign is called left hand side of the
equation and the expression on the right is called its right hand side. For
example, in the equation 5x Ð 4 = 3x + 8. 5x Ð 4 is the left hand side, 3x + 8 is
the right hand side and x is the variable or, the unknown quantity. In the above
equation, the degree of x is 1; it is a simple equation. If there is only one
unknown quantity with first degree in an equation, then the equation is called an
equation of first degree or a simple equation. The highest degree of x in the
equation x^2 Ð 4x = x Ð 6 is 2. It is a quadratic equation.

If there is a single variable of second degree in an equation, then it is called an
equation of second degree or a quadratic equation.

Some postulates are used to solve an equation. For example:

**Postulate 1**: If equal quantities are added to equal quantities, then the sums are
equal.

**Postulate 2**: If equal quantities are subtracted from equal quantities, then the
differences are equal.

**Postulate 3**: If equal quantities are multiplied by equal numbers, then the
products are equal.

**Postulate 4**: If equal quantities are divided by non zero equal numbers, then the
quotients are equal.

Besides these postulates, some rules are followed in finding the values of the
unknown in the equation.

(i) All terms containing the unknown in the equation are kept in the left hand
side.

(ii) To transfer any term from the left hand side to the right hand side or from
the right hand side to the left hand side, the sign before the term is to be
changed. This is called the method of transposition. Actually, it is an
application of the Postulate 2.
Mathematical Open Sentences related to Single Variable

(iii) If the equation is of the form \( \frac{a}{b} = \frac{c}{d} \), then \( ad = bc \) [multiplying both the sides by \( bd \)]. The products of the denominator of one side and the numerator of the other side are equal. It is called cross multiplication. This rule also holds good if one side is a fraction and the other side is an integer. Because, any integer can be considered as a fraction, with denominator 1.

For example, \( c = \frac{c}{1} \) and if \( \frac{a}{b} = c \), then \( \frac{a}{b} = \frac{c}{1} \) or \( a = bc \).

Conversely, if \( bd \neq 0 \) and \( ad = bc \), then \( \frac{a}{b} = \frac{c}{d} \).

Using the above rules once or more than once, if an equation is transferred to another equation then the equation thus obtained is equivalent to the given equation. Any simple equation can be reduced to the form \( ax = b \) by this process. If \( a \neq 0 \), then the root of the last equation can be obtained in the form \( x = \frac{b}{a} \). Some examples of solving equations are given below:

**Example 1.** Solve : \( \frac{y}{a} + a = \frac{y}{b} + b \) [where \( a \neq b \)]

**Solution:** Given, \( \frac{y}{a} + a = \frac{y}{b} + b \) \( \therefore \) \( \frac{y}{a} - \frac{y}{b} = b - a \) [by transposition]

or, \( \frac{y}{ab} = b - a \)

\( \therefore \) \( \frac{y}{ab} = 1 \) [dividing both sides by \( b \neq a \neq 0 \)]

or, \( y = ab \)

\( \therefore \) the required solution : \( y = ab \)

If the numerators of two functions are equal but their denominators are unequal and the values of the fractions are equal then the numerators must be zero. If this principle is used, then sometimes the solution procedure becomes very simple. Let us observe the following example:

**Example 2.** Solve : \( \frac{1}{x + 2} + \frac{1}{x + 5} = \frac{1}{x + 4} + \frac{1}{x + 3} \)

**Solution:** Given that,

\( \frac{1}{x + 2} + \frac{1}{x + 5} = \frac{1}{x + 4} + \frac{1}{x + 3} \) or, \( \frac{x + 5 + x + 2}{(x + 2)(x + 5)} = \frac{x + 3 + x + 4}{(x + 4)(x + 3)} \)

or, \( \frac{2x + 7}{x^2 + 7x + 10} = \frac{2x + 7}{x^2 + 7x + 12} \)
The values of the fractions are equal, their numerators are equal but denominators are unequal.
Hence, \(2x + 7 = 0\), or, \(2x = \frac{7}{2}\). ∴ \(x = \frac{7}{2}\)
∴ the required solution : \(x = \frac{7}{2}\)

**Example 3.** Solve : \(2z + \sqrt{2} = 3z - 4 \sqrt{2}\)

**Solution:** Given \(2z + \sqrt{2} = 3z - 4 \sqrt{2}\)
Hence, \(2z - 3z = -4 \sqrt{2}\) [by transposition]
or, \(z = 4 + 4 \sqrt{2}\)
or, \(z = 4(1 + \sqrt{2})\)
∴ the required solution : \(z = 4(1 + \sqrt{2})\)

Various techniques are often used in solving equations involving fractions. A few example of such techniques are given below. The success in solving equations depends mainly on experiences and practice.

**Example 4.** Solve : \(\frac{6x + 1}{15} - \frac{2x - 4}{7x - 1} = \frac{2x - 1}{5}\)

**Solution:** Given,
\[
\frac{6x + 1}{15} - \frac{2x - 4}{7x - 1} = \frac{2x - 1}{5}
\]
∴ \(\frac{6x + 1}{15} - \frac{2x - 4}{7x - 1} = \frac{2x - 1}{5}\) [by transposition]
or, \(\frac{6x + 1}{15} = \frac{2x - 4}{7x - 1}\) or, \(\frac{4}{15} = \frac{2x - 4}{7x - 1}\)
or, \(15(2x - 4) = 4(7x - 1)\) [by cross multiplication]
or, \(30x - 60 = 28x - 4\)
or, \(30x - 28x = 60 - 4\)
or, \(2x = 56\). ∴ \(x = \frac{56}{2} = 28\)
∴ the required solution set : \(\{28\}\)

Sometimes equations of the quadratic form are expressed into simple form and then the solution set can be determined.

**Example 5.** Find the solution set of \(\frac{2}{t - 1} + \frac{3}{t + 1} = \frac{5}{t}\)
Solution: \[2t (t + 1) + 3t (t \equiv 1) = 5 (t \equiv 1) (t + 1)\]

[Multiplying both sides by the L. C. M. of \((t \equiv 1), (t + 1), t\)]

or, \[2t^2 + 2t + 3t^2 \equiv 3t = 5t^2 \equiv 5\]

or, \[5t^2 \equiv t = 5t^2 \equiv 5\]

or, \[t = 5\]

\[\therefore \text{the required solution set: } S = \{5\}.\]

Exercise 6.1

Solve (Q. 1 \equiv 10):

1. \[5x \equiv 3 = 2x + 9\]

2. \[\frac{ax}{b} \equiv \frac{bx}{a} = a^2 \equiv b^2\]

3. \[\frac{1}{x + 1} + \frac{1}{x + 4} = \frac{1}{x + 2} + \frac{1}{x + 3}\]

4. \[\frac{1}{x \equiv 3} + \frac{1}{x \equiv 4} = \frac{1}{x \equiv 2} + \frac{1}{x \equiv 5}\]

5. \[\sqrt{3x \equiv 2} = 2\sqrt{3} + 4\]

6. \[(\sqrt{5} + 5)y + 4 = 9 + 5\sqrt{5}\]

7. \[\frac{2z \equiv 6}{9} + \frac{15 \equiv 2z}{12 \equiv 5z} = \frac{4z \equiv 15}{18}\]

8. \[\frac{x \equiv a}{b} + \frac{x \equiv b}{a} + \frac{x \equiv 3a}{a + b} = 0\]

9. \[\frac{a}{x \equiv a} + \frac{b}{x \equiv b} = \frac{a + b}{x \equiv a \equiv b}\]

10. \[\frac{4}{2x + 1} + \frac{9}{3x + 2} = \frac{25}{5x + 4}\]

Find the solution set (Q. 11 \equiv 20):

11. \[\frac{x + a}{x \equiv b} = \frac{x + a}{x + c} [b + c \neq 0]\]

12. \[\frac{x \equiv a}{a^2 \equiv b^2} = \frac{x \equiv b}{b^2 \equiv a^2}\]

13. \[\frac{x + a^2 + 2c^2}{b + c} + \frac{x + b^2 + 2a^2}{c + a} + \frac{x + c^2 + 2b^2}{a + b} = 0\]

14. \[\frac{x \equiv 2}{x \equiv 1} = 2 \equiv \frac{1}{x \equiv 1}\]

15. \[x(x^2 + 1) = 2x^2 + 2\]

16. \[\frac{x}{x \equiv 2} = 3\]

17. \[\frac{p}{p \equiv x} + \frac{q}{q \equiv x} = \frac{p + q}{p + q \equiv x}\]

18. \[\frac{1}{z} + \frac{1}{z + 1} = \frac{2}{z \equiv 1}\]

19. \[\frac{2z \equiv 1}{2z + 1} = \frac{3z \equiv 1}{3z + 2}\]

20. \[\sqrt{2x \equiv 3} + 5 = 2\]
The difference between an equation and an identity

\( c^2 + d(2c + d) = c(c + 2d) + d^2 \) is an identity. The expressions on the two sides, though appear to be different, are indeed the same. The values of both sides are equal for any values of \( c \) and \( d \). In an equation both sides are equal for some (one or more) particular values of the unknowns. But in an identity, both sides are equal for all values of the unknowns. The formulae in algebra are identities.

The use of simple equations

The variables (letters) used in equations denote numbers, not quantities. So we say, "Let the height of the tree be \( x \) metres or the number of the students be \( x \)." We do not say, "Let the height of the tree be \( x \)."

The algebraic process of solving a problem may be divided into the following stages.

(i) The variables (letters) are assumed to represent the required numbers.

(ii) Each statement is expressed according to the questions using the variables wherever applicable.

(iii) Equations are formed by connecting different parts of the question. The equations may be of first degree or second degree. The answer is obtained by solving the equations.

Example 6. A man requires one hour and a half to reach place B from A by car. The distance between the places is 96 km. On the way, some parts of the road was sloppy; the speed of the car in that part was 72 km/hour and for the rest of the way, it was 48 km/hour. How many km of the road was sloppy?

Solution: Let the length of the sloppy part of the road is \( x \) km. The length of the remaining part of road is then \( 96 - x \) km.

At the rate of 72 km/hour, the time to cover \( x \) km is \( \frac{x}{72} \) hours.

" " " " 48 " " " " " " " " " " (96 Ð x) km is \( \frac{96 - x}{48} \) hours

According to the problem, \( \frac{x}{72} + \frac{96 - x}{48} = \frac{3}{2} \) \[ \therefore 1 \frac{1}{2} = \frac{3}{2} \]

L. H. S. = \( \frac{2x + 3(96 - x)}{144} = \frac{2x + 288 - 3x}{144} = \frac{288}{144} \)

Hence, \( \frac{288}{144} = \frac{3}{2} \) or, \( \frac{288}{72} = 3 \) \[ \text{[multiplying both the sides by 2]} \]

or, \( 3 \times 72 = 288 \) or, \( x = 288 \) or, \( x = 72 \)

Answer: 72 km. of the road was sloppy.
Example 7. In a workshop the daily wages of a skilled labourer is Tk. 150/- and that of an unskilled labourer is Tk. 120/-. If the total number of labourer is 400 and the total daily wages amount of Tk. 52,800/, then find the number of skilled labourers.

Solution. Let the number of skilled labourers be x.
Then the number of unskilled labourers is 400 Ð x.
The total daily wages of skilled labourers amount to Tk. 150x.
" " " " " unskilled " " " " " Tk. 120 (400 Ð x)
According to the problem, 150x + 120 (400 Ð x) = 52800
or, 15x + 12 (400 Ð x) = 5280 [dividing both the sides by 10]
or, 15x + 4800 Ð 12x = 5280 or, 3x = 5280 Ð 4800
or, 3x = 480
or, x = \(\frac{480}{3}\) = 160.

Answer : Number of skilled labourers = 160.

Example 8. The difference of the digits in a number of two digits 2; the number obtained by interchanging the digits is 6 less than twice the given number. What is the number?

Solution : In this case, the value of the number is increased when the positions of the digits are interchanged, so the digit in the ones place is greater than that in the tens place.
Let, the digit in the tens place = x.
∴ the digit in ones place = x + 2.
∴ the number = 10x + (x + 2) = 11x + 2
The number obtained by interchanging the place is 10 (x + 2) + x = 11x + 20.
According to the problem.

\[2(11x + 2) \div 6 = 11x + 20\] or, \[22x + 4 \div 6 = 11x + 20\]
or, \[22x \div 11x = 20 + 2\] or, \[11x = 22\] or, \[x = \frac{22}{11} = 2\]
∴ the digit in the tens place of the number is 2;
hence the digit in ones place is 2 + 2 = 4
Answer : The number is 24.
Exercise 6.2

1. One number is \( \frac{2}{3} \) times of another number. If the sum of the two numbers is 100, find them.

2. If the same number be added to the numerator and the denominator of \( \frac{3}{5} \), then it changes to the fraction \( \frac{4}{5} \). What is that number?

3. The difference of numerator and denominator of a proper fraction is 1, and the fraction formed by subtracting 2 from the numerator and adding 2 with denominator is equal to \( \frac{1}{6} \); find the fraction.

4. The total number of passengers of a launch is 47. The fare per head for the cabin is twice that for the deck. The fare per head for the deck is Tk 30. If the total fare collected is Tk. 1680, then what is the number of passengers in the cabin?

5. The angle A of a triangle ABC is equal to the sum of other two angles. If the ratio of the measures of angles A and B is \( \frac{9}{4} \), then what is the measure of the angle C?

6. The digit in the tens place of a number consisting of two digits is twice the digit in the ones place. Show that the number is seven times the sum of the digits.

7. The sum of the digits of a number consisting of two digits is 9; the number obtained by interchanging the digits is 45 less than the given number. Find the number.

8. If 120 coins of twenty five paisa and ten paisa together be Tk. 27, then what is the number of each kind of the coins?

9. A man covered some distance at the rate of 60 km/hour and the rest at the rate of 40 km/hour while travelling in a car. He covered a total distance of 240 km in 5 hours. How far did he travel with 60 km/hour?

10. 3 benches in a class room remain vacant if 4 students sit in each bench. But 6 students are to remain standing if 3 students sit in each bench. What is the number of students in that class?
11. If the difference of squares of two consecutive numbers is 199, then what is the greater number?

12. A man invested a portion of Tk. 5600 at simple profit of 5% and the rest at simple profit of 4%. At the end of year he got a profit of Tk. 256. How much money did he investe at 5%?

**Inequality**

Postulates or rules related to equations are also applicable to inequality. With the exception that if the unequal quantities are multiplied or divided by equal negative numbers then the direction of inequality is reversed.

Let us consider the inequality \(4 < 6\).

\[
\therefore 4 + 2 < 6 + 2 \text{ or } 6 < 8. \quad \text{[adding 2 to both sides]}
\]

Similarly \(2 < 4\) \[
\therefore 8 < 12 \quad \text{[subtracting 2 from both sides]}
\]

\[
\therefore 2 < 3 \quad \text{[multiplying both sides by 2]}
\]

\[
\therefore 6 < 8. \quad \text{[dividing both sides by 2]}
\]

Multiplying both sides of the inequality by \(\mathcal{D} 2\), we get \(\mathcal{D} 8\) and \(\mathcal{D} 12\) separately.

Here, \(\mathcal{D} 8 > \mathcal{D} 12\).

Similarly, \(\mathcal{D} 2 > \mathcal{D} 3\). \[\text{[dividing both sides by } \mathcal{D} 2\]}

Generally it may stated that if \(a < b\), then

\[
a + c < b + c \quad \text{for all values of } c
\]

\[
a \mathcal{D} c < b \mathcal{D} c \quad \text{" } \text{ " } \text{ " } \text{ " } c
\]

\[
ac < bc \quad \text{for positive values of } c
\]

\[
\frac{a}{c} < \frac{b}{c} \quad \text{" } \text{ " } \text{ " } \text{ " } c
\]

\[
ac > bc \quad \text{for negative values of } c
\]

\[
\frac{a}{c} > \frac{b}{c} \quad \text{" } \text{ " } \text{ " } \text{ " } c
\]

**Example 9.** Solve and show the solution set on a number line : \(3x + 4 > 16\).

**Solution :** Given that, \(3x + 4 > 16\)

\[
\therefore 3x + 4 \mathcal{D} 4 > 16 \mathcal{D} 4 \quad \text{[subtracting 4 from both sides ]}
\]

or, \(3x > 12\)

or, \(\frac{3x}{3} > \frac{12}{3}\) \[\text{[dividing both sides by 3]}

or, \(x > 4\).

\[
\therefore \text{the required solution : } x > 4
\]

Here the solution set \(S = \{ x \in \mathbb{R} : x > 4 \} \).
The solution set is shown in the number line below. All real numbers greater than 4 are solutions of the given inequality.

Example 10. Solve and show the solution set on a number line: \( x \geq 9 > 3x + 1 \).

Solution: Given that, \( x \geq 9 > 3x + 1 \) \( \therefore x \geq 9 + 9 > 3x + 1 + 9 \)

or, \( x > 3x + 10 \) or, \( x \geq 3x > 3x + 10 \geq 3x \)

or, \( 2x > 10 \)

or, \( 2x \geq 10 \) [the direction of inequality is reversed on division of both sides by negative integer \( 2 \)]

or, \( x < 5 \).

\( \therefore \) the required solution: \( x < 5 \).

The solution set \( S = \{ x \in R : x < 5 \} \).

All real numbers less than 5 are the solutions of the given inequality.

N. B. The solution of an inequality are usually expressed by an inequality like solution of an equation is expressed by an equation (equality). The solution set of an inequality is usually an infinite subset of the set of real numbers. \( a \geq b \) means \( a > b \) or \( a = b \). i. e. \( a \geq b \) is false only if \( a < b \).

Therefore, \( 4 > 3 \) and \( 4 \geq 4 \) are correct statements.

Example 11. Solve: \( a(x + b) < c \), \( a \neq 0 \)

Solution: If \( a \) is positive, then \( \frac{a(x + b)}{a} < \frac{c}{a} \) [dividing both sides by \( a \)]

or, \( x + b < \frac{c}{a} \) or, \( x < \frac{c}{a} \geq b \)
If $a$ is negative, then by the same process we get, \( \frac{a(x + b)}{c} > \frac{c}{a} \)

or, \( x + b > \frac{c}{a} \)

or, \( x > \frac{c}{a} - b \)

\[ \therefore \] Required solution:

(i) \( x < \frac{c}{a} - b \) if \( a > 0 \)

(ii) \( x > \frac{c}{a} - b \) if \( a < 0 \)

N. B. If \( a = 0 \) and \( c \) is positive, then the inequality holds for any value of \( x \). But if \( a = 0 \) and \( c \) is negative or zero, then the inequality has no solution.

**Exercise 6.3**

Solve the inequalities and show the solution set on a number line:

1. \( y > 3 < 5. \)
2. \( 3 (x > 2) < 6 \)
3. \( 3x > 2 > 2x \)
4. \( z \leq \frac{1}{2} z + 3. \)
5. \( 8 \geq 2 \geq 2x \)
6. \( x \leq \frac{x}{3} + 4 \)
7. \( 5 (3 > 2t) \leq 3(4 > 3t) \)
8. \( \frac{x}{3} + \frac{x}{4} + \frac{x}{5} > \frac{47}{60} \)

**Application of Inequalities:**
We have discussed problem solving using equations. We now consider some problems involving inequalities.

**Example 12.** In an examination, Tina obtained \( 5x \) and \( 6x \) marks and Kumkum obtained \( 4x \) and 84 marks in Bangla first and second paper respectively. None of them secured less than 40 marks. Thus Kumkum secured the first position and Tina secured the second position in Bangla. Express the possible values of \( x \) using inequalities.

**Solution:** The total marks secured by Tina and Kumkum in Bangla are \((5x + 6x)\) and \((4x + 84)\) respectively.

According to the problem

\( 5x + 6x < 4x + 84 \) or, \( 5x + 6x > 4x < 84 \)

or, \( 7x < 84 \) or, \( x < \frac{84}{7} \)

or, \( x < 12 \)

Moreover, \( 3x \geq 40 \). \[ \therefore 4x \text{ is the lowest number} \]

**Answer:** \( 10 \leq x \leq 12 \).
Example 13. A student has bought $x$ pencils at Tk. 5 each and $(x + 4)$ khatas at Tk. 8 each. If the total cost does not exceed Tk. 97, what is the maximum number of pencils he has bought?

Solution: The price of $x$ pencils is Tk. 5$x$ and that of $(x + 4)$ khatas is Tk. 8$(x + 4)$.

According to the problem,

$$5x + 8(x + 4) \leq 97$$

or,

$$5x + 8x + 32 \leq 97$$

or,

$$13x \leq 65$$

or,

$$x \leq \frac{65}{13}$$

or,

$$x \leq 5.$$  

Answer: The maximum number of pencil the students has bought is 5.

Exercise 6.4

Express the problems 1Ð5 in terms of inequalities and find the possible values of $x$:

1. A boy walked 3 hours at the rate of 3 km/hour and run $\frac{1}{2}$ hour at the rate of $(x + 2)$ km/hour, and the distance covered by him was less than 29 km.

2. A boarding house requires $4x$ kg. of rice and $(x - 3)$ kg of pulses every day and it does not require more than 40 kg. of rice and pulses together.

3. Mr. Sohrab bought $x$ kg. mango at the rate of Tk. 30 per kg. He gave a Tk. 500 note to the seller. The seller returned the balance which included $x$ notes of Tk. 20.

4. A car runs $x$ km. in 4 hours and $(x + 20)$ km. in 5 hours. The average speed of the car does not exceed 100 km/hour.

5. The area of a piece of paper is 40 sq. cm. A rectangular piece $x$ cm. long and $t$ cm. wide is cut off from it.

6. The ages of the son is one-third that of the mother. The father is 6 years older than the mother. The sum of the ages of these three is not more than 90 years. Express the age of the father by means of an inequality.

7. Nadira appeared at junior scholarship examination at the age of 14 years. She will appear at the S. S. C. examination at the age of 17 years. Express her present age by means of an inequality.
8. The speed of a jet-plane does not exceed 300 metres/sec. Express the time (in seconds) required by the plane to cover 15 km. in the form of an inequality.

9. The air distance of Jedda from Dhaka is 5000 km. The maximum speed of a jet-plane is 900 km/hour; but on way from Dhaka to Jedda, it faces air flowing of 100 km/hour from opposite direction. Express the time required for the non-stop flight from Dhaka to Jedda in terms of an inequality.

10. On the basis of above problem, express the time required for the non-stop flight from Jedda to Dhaka in the form of an inequality.

11. 5 times a positive integer is less than the sum of twice the number and 15. Express the possible values of the number in the form of an inequality.

**Quadratic Equation**

An equation of the form \( ax^2 + bx + c = 0 \) (where \( a \neq 0 \)) is called a quadratic equation. The left hand side of a quadratic equation is polynomial of second degree. It is to be noted that the right hand side of the equation has been taken to be zero. Its left hand side is a polynomial of second degree.

If \( \alpha \) is substituted for \( x \) in the expression \( f(x) = ax^2 + bx + c \) and its value \( f(\alpha) \) becomes zero, then \( \alpha \) is said to be a root of the equation \( ax^2 + bx + c = 0 \). For example, 3 is a root of the equation \( x^2 - 7x + 12 = 0 \), because \( 3^2 - 7 \cdot 3 + 12 = 0 \). An other root of this equation is 4, because \( 4^2 - 7 \cdot 4 + 12 = 0 \). Thus there are two roots of the equation \( x^2 - 7x + 12 = 0 \).

\( x = \mp 1 \) is the only solution of the equation \( x^2 + 2x + 1 = 0 \), because the left hand side = \((x + 1)^2\). On the other hand the equation \( x^2 + 2x + 2 = 0 \) does not have any solution in real numbers. Because, \( x^2 + 2x + 2 = (x + 1)^2 + 1 \) and as the square of any real number is always \( \geq 0 \) the value of \( x^2 + 2x + 2 \) cannot be zero for real values of \( x \). Hence a quadratic equation may have two roots or one root only, it may not have any solution at all. But it is a fact that a quadratic equation cannot have more than two roots. Here equations whose left hand side can be resolved into factors will be considered, as such equations have solutions in real numbers. In solving such equations, the following important property of real numbers is used.
There is an important property for solution by resolving into factors.

The product of two non zero numbers cannot be zero. In other words, if the product of two numbers is zero then at least one of them must be zero. That is for any real values of $a$, $b$; $ab = 0$ if and only if $a = 0$ or $b = 0$.

**Example 14.** Find the solution set of $(x \div 3) (x + 2) = 0$.

**Solution:** If $(x \div 3) (x + 2) = 0$, then either $x \div 3 = 0$ or $x + 2 = 0$.

Hence, $x = 3$ or, $x = \div 2$

$\therefore$ the required solution set is $\{3, \div 2\}$.

**Example 15.** Find the solution set of $y^2 = \sqrt{2}y$.

**Solution:** Given that, $y^2 = \sqrt{2}y$. or, $(y^2 \div \sqrt{2}y) = 0$ [Making R. H. S. zero]

or, $y (y \div \sqrt{2}) = 0$

$\therefore y = 0$ or, $y \div \sqrt{2} = 0$

i. e. $y = 0$ or, $y = \sqrt{2}$

$\therefore$ the required solution set is $\{0, \sqrt{2}\}$

**Example 16.** Find the solution set of $\frac{x \div 2}{x + 2} + \frac{6(x \div 2)}{x \div 6} = 1$

**Solution:** From the given equation.

$\frac{6(x \div 2)}{x \div 6} = 1 \div \frac{x \div 2}{x + 2}$

or, $\frac{6(x \div 2)}{x \div 6} = \frac{x + 2}{x + 2} = \frac{4}{x + 2}$

or, $\frac{6(x \div 2)}{x \div 6} = \frac{4}{x + 2}$ or, $\frac{3(x \div 2)}{x \div 6} = \frac{2}{x + 2}$

or, $3(x \div 2) (x + 2) = 2 (x \div 6)$ [by cross multiplication]

or, $3x^2 \div 4 = 2(x \div 6)$ or, $3x^2 \div 12 = 2x \div 12$

or, $3x^2 \div 2x \div 12 + 12 = 0$ or, $3x^2 \div 2x = 0$

or, $x(3x \div 2) = 0$

$\therefore x = 0$ or, $3x \div 2 = 0$

That is, $x = 0$ or, $x = \frac{2}{3}$

$\therefore$ the required solution set is $\{0, \frac{2}{3}\}$
Exercise 6.5

Find the solution set of the following equations:

1. \((x + 1) (x + 2) = 0\)
2. \((x + 3) (x \sqrt{5}) = 0\)
3. \((\sqrt{2p} \cdot 3) (\sqrt{2p} + \sqrt{5}) = 0\)
4. \(2(z^2 \cdot 9) + 9z = 0\)
5. \(v(v \cdot 10) = v \cdot 10\)
6. \(12(x^2 + 1) = 25x\)
7. \(\frac{3}{2x + 1} + \frac{4}{5x} = 1 = 2\)
8. \(\frac{x + 7}{x + 1} + \frac{2x + 6}{2x + 1} = 5\)
9. \(\frac{3}{q} + \frac{4}{q + 1} = 2\)
10. \(\frac{x \cdot a}{x \cdot b} + \frac{x \cdot b}{x \cdot a} = \frac{a + b}{a}\)
11. \(\frac{4}{\sqrt{10x} \cdot 4} + \sqrt{10x} \cdot 4 = 5\)
12. \((x + 5) (x \cdot 5) = 24\)
13. \(\frac{x + a}{x} = \frac{x}{b} + \frac{b}{x}\)
14. \(\frac{ax + b}{a + bx} = \frac{cx + d}{c + dx}\)
15. \(\frac{1}{a + b + x} = \frac{1}{a} + \frac{1}{b} + \frac{1}{x}\)
16. \(\left(\frac{x + a}{x \cdot a}\right)^2 \cdot 5 \left(\frac{x + a}{x \cdot a}\right) + 6 = 0\)
17. \(\frac{(x + 1)^3}{(x + 1)^2} = 2\)
18. \(x + \frac{1}{x} = 2\)
19. \(x \cdot 4 = \frac{x \cdot 4}{x}\)
20. \(2x^2 \cdot 8ax = 0\)

Application of Quadratic Equations

We now consider some problems which are solved by forming quadratic equations from the given conditions.

Example 17. Which natural number when added to its square makes the sum equal to 9 times the next natural number. What is the number?

Solution: Let the required natural number be \(x\). Then the next number = \(x + 1\). According to the problem,

\[x^2 + x = 9(x + 1)\]

or, \(x^2 + x = 9x + 9\)

or, \(x^2 + x \cdot 9x \cdot 9 = 0\)

or, \((x + 1) (x \cdot 9) = 0\)

Hence, \(x + 1 = 0\) or, \(x \cdot 9 = 0\)

\(\therefore \) \(x = 1\) or, \(x = 9\).

But \(1\) is not a natural number. Hence the required number is 9.
**Example 18.** In a proper fraction the denominator is greater than numerator by 4. The denominator of the fraction obtained by squaring this fraction is greater than the numerator by 40. Find the fraction.

**Solution:** Let the numerator of the required fraction be \( x \).

\[ \therefore \quad \text{then its denominator} = x + 4. \]

\[ \therefore \quad \text{the fraction} = \frac{x}{x + 4} \quad \text{and its square} = \frac{x^2}{(x + 4)^2} = \frac{x^2}{x^2 + 8x + 16} \]

According to the problem,

\[ x^2 + 8x + 16 = x^2 = 40 \quad \text{or,} \quad 8x = 24 \quad \text{or,} \quad x = 3. \]

\[ \therefore \quad \text{the required fraction is} \quad \frac{x}{x + 4} = \frac{3}{7}. \]

**Example 19.** The hypotenuse of a right-angled triangle is 15 cm. and the difference of other two sides is 3 cm. Find the length of those sides.

**Solution:** Let the length of the smallest side of the triangle = \( x \) cm. and the length of the other side = \( x + 3 \) cm.

By the theorem of Pythagorus.

\[ x^2 + (x + 3)^2 = 15^2 \]

or, \[ x^2 + x^2 + 6x + 9 = 225 \]

or, \[ 2x^2 + 6x \not\equiv 216 = 0 \]

or, \[ 2(x^2 + 3x \not\equiv 108) = 0 \]

or, \[ x^2 + 3x \not\equiv 108 = 0 \]

or, \[ x(x + 12) \not\equiv 9 (x +12) = 0 \]

or, \[ (x + 12) (x \not\equiv 9) = 0 \]

\[ \therefore \quad x + 12 = 0 \quad \text{or,} \quad x \not\equiv 9 = 0 \]

\[ \therefore \quad x = 12 \quad \text{or,} \quad x = 9 \]

As the length can not be negative.

Thus the length of the smallest side of the triangle = 9 cm. and the length of the other side = \((9 + 3)\) cm. = 12 cm.
Exercise 6.6

1. The length of a rectangular region is 4 metre longer than its breadth. If its area is 192 sq. metres, then what is its perimeter?

2. Find a positive number which is 72 less than its square.

3. The denominator of a proper fraction is greater than its numerator by 2. The denominator of the fraction obtained by squaring this fraction is greater than its numerator by 48. Find the fraction.

4. The area of a rectangular room is 192 sq. metres. If its length is decreased by 4 metres and breadth is increased by 4 metres, then the area remains unchanged. What is the length of the room?

5. The base of a triangular region is 6 metres greater than twice its height. If the area of the region is 810 sq. metres, then what is its height?

6. There is a road of equal breadth inside of rectangular garden 50 metres long and 40 metres wide. If the area of the garden without the road is 1200 sq. metres, then what is the breadth of the road?

7. Shahnewaz bought a rickshaw at Tk. 6000 and sold it to Yusuf at a profit of \(x\)% . Yusuf sold it again to Sohel at a profit of \(x\)% . The cost price of Sohel is Tk. 2640 more than that of Shahnewaz. Find the value of \(x\).

8. The sum of digits of a number of two digits is 12 and the product of the digits of the number is 32. What is the number?

9. A man bought some pens at Tk. 240 and observed that if he would have got one more pen, then the price of each pen would have been less by Tk. 1 on the average. How many pens did he buy?

10. The perimeter of a rectangular region is 64 metres and its area is 231 sq. metres. Find the length and breadth of the region.

11. The hypotenuse of right-angled triangle is 13 cm. and its perimeter is 30 cm. What is the area of triangular region?

12. In a right-angled triangle, the sides adjacent of the right angle are \(x\) metres and \((x + 3)\) metres long. The area of the triangular region is 170 sq. metres. What is the value of \(x\)?
13. The length of the perpendicular from the centre of a circle to a chord is 2 cm. less than half the length of the chord. If the radius of the circle is 10 cm. then what is the length of the chord?

14. The sum of the marks obtained by x students in Mathematics is 1190. When the mark of a student who got 88 in Mathematics is added to this sum, the average marks of the students increased by 1. What is the value of x?

15. Student of a class collected Tk. 70 by contributing 30 paisa more than the number of paisa equal to the number of students in the class. What is the number of the students in that class?

**Quadratic Inequality**

In solving quadratic equations, the following property of real numbers was used, \(ab = 0\) if and only if \(a = 0\) or \(b = 0\).

In solving quadratic inequalities, the following property or real numbers will be needed; \(ab > 0\) if and only if \(a\) and \(b\) are both positive or both negative. The method of solution will be clear from the examples discussed below.

**Example 20.** Solve and indicate the solution set in number line:

\[(x +1) \ (x - 3) > 0\]

**Solution**: Here we are required to find those values of \(x\) for which the inequality holds good. The product of two factors will be positive if and only if the factors are both positive or both negative.

Hence \((x + 1) \ (x - 3) > 0\) if and only if \(x + 1\) and \(x - 3\) are both positive or both negative.

Now, \(x + 1 < 0\) when \(x < -1\) and \(x + 1 > 0\) when \(x > -1\); \(x - 3 < 0\) when \(x < 3\) and \(x - 3 > 0\) when \(x > 3\).

\[
\therefore \; \; (x + 1) \; \text{and} \; (x - 3) \; \text{are both positive only when} \; x > 3, \; (x + 1) \; \text{and} \; (x - 3) \; \text{are both negative only when} \; x < -1.
\]

Hence \((x + 1) \ (x - 3) > 0\) if and only if \(x < -1\) or \(x > 3\).

\[
\therefore \; \; \text{the required solution :} \; x < -1 \; \text{or} \; x > 3
\]

Hence, solution set is \(\{x \in \mathbb{R} : x < -1 \; \text{or} \; x > 3 \} \).
Example 21. Solve and find the solution set on a number line:

\[ x^2 - 3x + 2 < 0. \]

**Solution:** Given that, \( x^2 - 3x + 2 < 0. \)

Now, \( x^2 - 2x - x + 2 = x(x - 2) - 1(x - 2) = (x - 2)(x - 1) \)

Hence the inequality becomes \( (x - 2)(x - 1) < 0. \)

Now \( (x - 2)(x - 1) < 0 \) if and only if \((x - 2)\) and \((x - 1)\) one is positive and the other is negative.

If \( x < 1, \) then \( x - 1 < 0, \) \( (x - 2) < 0 \)

If \( 1 < x < 2, \) then \( x - 1 > 0, \) \( x - 2 < 0 \)

If \( x > 2, \) then \( x - 1 > 0, \) \( x - 2 > 0 \)

\( \therefore \) the required solutions: \( 1 < x < 2 \)

\( \therefore \) the solution set is \( \{x \in \mathbb{R} : 1 < x < 2 \} \)

This is shown on the number line:

**Exercise 6.7**

Solve the following inequalities and show the solution on number lines:

1. \((x - 2)(x - 3) > 0. \)
2. \((x - 1)(x + 2) \geq 0. \)
3. \((2x - 1)(x + 2) > 0. \)
4. \((x^2 - 2x + 1) > 0. \)
5. \(x^2 - 6x - 7 > 0. \)
6. \(x^2 - 2x - 15 > 0. \)
7. \(x^2 - 8x + 15 > 0. \)
8. \(x^2 - 9x + 8 \leq 0. \)
9. \((5x - 6)(x - 3) < 0. \)
10. \(2x^2 - 3x + 1 < 0. \)

**Application of Quadratic Inequalities**

A few mathematical problems related to quadratic inequalities are considered below:

**Example 22.** The difference of two natural numbers is 2 and product of the numbers is greater than 14. What are the least possible numbers?
Solution: Let the smaller number = x.
\[ \therefore \quad \text{The greater number} = x + 2 \]
\[ \therefore \quad x(x + 2) > 14 \]

According to the problem, the product is at least \(14 + 1 = 15\).

Let, \(x(x + 2) = 15\)

or, \(x^2 + 2x = 15\)

or, \(x^2 + 2x - 15 = 0\)

or, \((x + 5)(x - 3) = 0\)

Hence, \(x + 5 = 0\) or, \(x - 3 = 0\) i.e., \(x = -5\) or, \(x = 3\)

But \(x = -5\) is not acceptable.

\[ \therefore \quad x = 3 \quad \text{and} \quad x + 2 = 3 + 2 = 5. \]

\[ \therefore \quad \text{The least possible numbers are 3 and 5.} \]

Example 23. The product of the two consecutive natural numbers is greater than 89. What are the least possible numbers?

Solution: Let the smaller number = x  \[ \therefore \quad \text{the other number} = x + 1 \]

According to the problem, \(x(x + 1) > 89\)

Let, \(x(x + 1) = 90\)

or, \(x^2 + x - 90 = 0\)

or, \((x + 10)(x - 9) = 0\)

\[ \therefore \quad x + 10 = 0 \quad \text{or} \quad x - 9 = 0 \quad \text{i.e.} \quad x = -10 \quad \text{or} \quad x = 9 \]

But \(x = -10\) is not acceptable.

\[ \therefore \quad x = 9 \quad \therefore \quad x + 1 = 9 + 1 = 10. \]

\[ \therefore \quad \text{The least possible numbers are 9 and 10.} \]
Exercise 6.8

1. The difference of two natural numbers is 9 and their product is greater than 9. Find the two least possible numbers.

2. The product of two consecutive even numbers is greater than 358. Express the problem by inequality and solving find what could be two such least possible numbers.

3. The product of two consecutive numbers is greater than 649. Express the problem in the form inequality and find what could be the two such least possible numbers.

4. The difference of two natural numbers is 5 and their product is greater than 12. Express the problem in terms of inequality and find what could be two such least possible numbers.

5. If 6 be added to the square of natural number less than 10, then the sum is 5 times greater than that number. Find the possible sets of the numbers.
Multiple Choice Questions (MCQ) :

1. Which one of the followings is an identity?
   A. $4ab = (a + b)^2 + (a - b)^2$
   B. $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$
   C. $(x + a)(x + b) = x^2 + (a + b)x + ab$
   D. $a^2 + 5a + 5 = 0$

2. If $\frac{y}{p} + p = \frac{y}{q} + q$, then which is the value of $y$?
   A. $pq$
   B. $p - q$
   C. $\frac{pq}{p - q}$
   D. $\frac{p + q}{pq}$

3. Which is the root of the equation $\frac{3x}{3} + \frac{4x}{4} + \frac{5x}{5} = 1$?
   A. $\frac{1}{17}$
   B. $1$
   C. $0$
   D. $17$

4. Given that:
   i. $2x + 3 = 9$
   ii. $\frac{x}{2} + 3 = 1$
   iii. $3x = 3$
   Which of the above equations are equivalent?
   A. i and ii
   B. ii and iii
   C. i and iii
   D. i, ii and iii

Answer questions no. (5 Ð 7) based on the following equation:
\[ \sqrt{3x} + 3 = 4 \]

5. Which one of the followings is the correct value of $x$?
   A. $\frac{1}{\sqrt{3}}$
   B. $4\sqrt{2}$
   C. $\frac{1}{3}$
   D. $3$

6. Which one of the following equations is equivalent to the given equation?
   A. $\sqrt{6x} + 2\sqrt{3} = 4\sqrt{2}$
   B. $x + \sqrt{3} = 4$
   C. $\sqrt{6x} = 3$
   D. $3\sqrt{x} + 3 = 4$
7. Which one of the following is correct in terms of postulate of the equations?
   A. \( \sqrt{3x} = 0 \)  
   B. \( \sqrt{3x} + 1 = 4 \)
   C. \( \sqrt{3x} + 1 = 2 \)  
   D. \( \sqrt{3x} = 1 \)

8. Sum of the numerator and denominator of a fraction is 5 and their difference is 1. What is the fraction?
   A. \( \frac{3}{2} \)  
   B. \( \frac{1}{4} \)
   C. \( \frac{2}{3} \)  
   D. \( \frac{4}{5} \)

9. Difference of two numbers is 4; square of the smaller number is equal to the double of the bigger number. What is the value of bigger number?
   A. 2  
   B. 4
   C. 6  
   D. 8

10. Given that; Shupti has Tk. \( x \) and Popy's taka is \( \frac{2}{3} \) times of Shupti's taka. If 3 times of the sum of their money is Tk. 18000, then the probable equations may be defined as
    
    i. \( x + \frac{2x}{3} = 18000 \)
    ii. \( x + \frac{2x}{3} = 6000 \)
    iii. \( 3 \left(x + \frac{2x}{3}\right) = 18000 \)

    Which one of the above equations are correct based on the given condition?
    A. i and ii  
    B. ii and iii
    C. i and iii  
    D. i, ii and iii

Let, one acute angle is twice of the other acute angle of a right angle triangle ABC.

Based on the above information, answer questions no. (11Ð13):

11. What is the ratio of the acute angles?
    A. 1 : 2  
    B. 2 : 1
    C. 1 : 3  
    D. 1 : 4

12. If the sum of the acute angles is \( 3x \), then what is the value of \( x \)?
    A. 90°  
    B. 60°
    C. 30°  
    D. 10°
13. What is the complementary angle of the sum of the acute angles?
A. 180°  B. 100°  C. 90°  D. 0°

14. If \(a < b\) and \(c > 0\), then which one of the following relations is correct?
A. \(\frac{a}{c} = \frac{b}{c}\)  B. \(\frac{a}{c} > \frac{b}{c}\)
C. \(\frac{a}{c} \geq \frac{b}{c}\)  D. \(\frac{a}{c} > \frac{b}{c}\)

15. What is the meaning of \(a < 0\) ?
A. \(a\) is a negative number  B. \(a\) is a real number
C. \(a\) is a positive number  D. \(a\) is an absolute value

16. If both sides of the inequality is divided by 3, then the inequality will be
A. \(\frac{1}{5} (3 \div 2x) \leq (4 \div 3x)\)
B. \(\frac{1}{5} (3 \div 2x) \leq \frac{1}{3} (4 \div 3x)\)
C. \(\frac{5}{3} (3 \div 2x) \leq (4 \div 3x)\)
D. \((3 \div 2x) \leq (4 \div 3x)\)

17. Which one of the following is the solution of the inequality?
A. \(x > 3\)  B. \(x \leq 3\)
C. \(x \geq \frac{1}{2} 3\)  D. \(x \geq 3\)

18. Which one of the following number lines represents the inequality?
A. 

B. 
C. 
D. 

19. How many roots are there in the equations \(ax^2 + bx + c = 0\) [\(a \neq 0\)]?
A. 1  B. 2
C. 3  D. 4
20. Hypotenuse and base of a right angle triangle are 5 unit and 3 unit respectively. How many unit is its altitude?
   A. 16  B. 9  
   C. 4  D. 2
21. The sum of a number and its reciprocal proportion is 2. The probable equation may beÑ
   i. \( x + \frac{1}{x} = 2 \)  
   ii. \( x^2 + 2x + 1 = 0 \)  
   iii. \( x^2 - 2x + 1 = 0 \)  
Which of the above equations are appropriate?
   A. i and iii  B. ii and iii  
   C. i and ii  D. i, ii and iii
22. If the digit in tens place is x, then what is the ones place digit?
   A. 3x  B. \( \frac{3}{x} \)  
   C. \( x + 3 \)  D. \( \frac{x}{3} \)
23. If the digit in ones place is 3, then what is the number?
   A. 13  B. 31  
   C. 39  D. 93
24. If the digit in tens place is 2, then what is the number after interchanging the digits?
   A. 62  B. 26  
   C. 21  D. 12
25. Your brother has Tk. 1 more than you and your sister has Tk. 3 less than you. If you have Tk. x, then the product of your brother and sister's taka can be defines as inequalityÑ
   A. \( x (x - 1) (x + 3) > 0 \)  B. \( x (x - 1) (x + 3) < 0 \)  
   C. \( (x - 1) (x + 3) > 0 \)  D. \( (x - 1) (x - 3) < 0 \)
Answer questions no. (26–28) based on the following information:
The difference of length and width of a rectangle is 2 unit and its area is more than 8 square unit.

26. The above problem can be defined in inequality as
   A. \( x (x + 2) + 8 > 0 \)
   B. \( x (x + 2) > 8 \)
   C. \( x (x + 2) \geq 8 \)
   D. \( 8 > x (x + 2) \)

27. Length is how many times of the width?
   A. Twice
   B. Half
   C. One-fourth
   D. Two-third

28. The probable set of length and width is
   A. \( \{3, 2\} \)
   B. \( \{2, 3\} \)
   C. \( \{4, 2\} \)
   D. \( \{2, 4\} \)

**Creative Questions:**

1. Sum of the digits of a two digit number is 7; if the digits are interchanged, then the new number is 9 more than the given number.
   A. Write the given number and the number interchanging the digits by using one variable.
   B. Determine the number.
   C. If the digits of the number denote the length and width of a rectangle, then what is the length of its diagonal? Then taking the length of the diagonal as a side of a square, find the length of the diagonal of that square.

2. There are 792 students now in the Probaho Biddhyaniketon School. Number of boy students is 58 more than that of number of girl students. Two years ago the number of students was 98 more than the \( \frac{3}{4} \) times of number of students having now.
   A. What is the number of girl students of that school now?
   B. Determine the ratio of number of boy and girl students two years ago.
   C. What are the numbers of girl and boy students if the square of the difference of the number of boy and girl student is 34 less than the one-fourth of the girl students?
3. The base and altitude of a right angle triangle are \((x - 1)\) cm and \(x\) cm respectively. Again, length of a side of a square is as same as the altitude of the triangle. The length and width of another rectangle are \((x + 13)\) cm and \((x + 3)\) cm respectively, \(x = 5\) unit.

A. Find the ratio of the area of the above three areas.

B. Determine the area of the rectangle, if the numerical value of the perimeter of the rectangle is 10 times of the area of the square.

C. Determine the area of the square, if the length of the hypotenuse of the triangle is 2 cm less than the width of the rectangle.
Chapter VII
Relation, Function and Graph

Relation: If A and B be two sets; then any non-zero subset R of ordered pairs which belong to the set of Cartesian product $A \times B$ is called a relation from A to B. When $x$ is an element of set A and $y$ is an element of set B and $(x, y) \in R$, then we write, $x \, R \, y$ and we read it as "$x$ is related to $y$" i.e., the element $x$ in R related to the element $y$.

A relation $R$ from $A$ to $A$ i.e. if $R \subset A \times A$, then $R$ is called a relation on $A$. In practice, generally when relation between two sets $A$ and $B$ and their elements are given, then the sets of those ordered pairs $(x, y)$ with $x \in A$, $y \in B$ so related is the concerned relation.

Example 1. If $A = \{3, 4\}$, $B = \{2, 3\}$ and we consider the condition $x > y$ amongst the elements $x \in A$ and $y \in B$, then what is the concerned relation?

Solution: According to the question, the relation is $R = \{(x, y) : x \in A, y \in B \text{ and } x > y\}$.

Hence, $A \times B = \{3, 4\} \times \{2, 3\} = \{(3, 2), (3, 3), (4, 2), (4, 3)\}$

∴ According to given condition, $R = \{(3, 2), (4, 2), (4, 3)\}$

Example 2. If $C = \{1, 4\}$, $D = \{3, 5\}$ and we consider the condition $x < y$ amongst the elements of $x \in C$ and $y \in D$, then what is the concerned relation?

Solution:
Here, $R = \{(x, y) : x \in C, y \in D \text{ and } x < y\}$

Hence, $C \times D = \{1, 4\} \times \{3, 5\} = \{(1, 3), (1, 5), (4, 3), (4, 5)\}$

∴ $R = \{(1, 3), (1, 5), (4, 5)\}$

Exercise 7.1
1. If $A = \{5, 6\}$, $B = \{4, 5\}$ and we consider the condition $x > y$ amongst the elements of $x \in A$ and $y \in B$, then describe the relation.

2. If $C = \{3, 4\}$, $D = \{2, 5\}$ and we consider the condition $x < y$ amongst the elements of $x \in C$ and $y \in D$, then describe the relation.
**Function**: If \( y = x^2 - 4x + 3 \), then \( y \) is a function of \( x \). Because, for each value of \( x \), there is a definite value of \( y \). Here the values of the variable \( y \) is dependent on the values of the variable \( x \).

**In general we can say that**: If two variables \( x \) and \( y \) are so related that for each value of \( x \), there is one and only one value of \( y \), then \( y \) is called function of \( x \).

Again, the circumference of a circle having radius \( r \) is \( c = 2\pi r \). Here \( c \) is a function of \( r \) and \( \pi \) is constant. If the value of \( r \) is increased or decreased, then the value of \( c \) will increase or decrease. That is, increase and decrease in the value of \( c \) is dependent on the increase and decrease in the value of \( r \).

**Interpretation of function with the help of sets**: Let \( X \) and \( Y \) be two nonempty sets. If a rule or formula \( f \) is given such that to each element \( x \) of the set \( X \) there is assigned a unique element \( y \) of \( Y \), then it is called a function from \( X \) into \( Y \). In such cases, we write, \( y = f(x) \).

**Example 3**. Let \( P = \{1, 2, 3, 4\} \) and \( Q = \{90, 80, 95, 60\} \). If \( P \) and \( Q \) be the sets of roll numbers of four students of a certain class and the set of marks obtained by these four students in mathematics respectively and if the elements of \( P \) and \( Q \) are placed in tabular form, the table will be as follows:

<table>
<thead>
<tr>
<th>Roll numbers</th>
<th>marks obtained</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>90</td>
</tr>
<tr>
<td>2</td>
<td>80</td>
</tr>
<tr>
<td>3</td>
<td>95</td>
</tr>
<tr>
<td>4</td>
<td>60</td>
</tr>
</tbody>
</table>

The above table describes a function from \( P \) into \( Q \). Here, \( f(1) = 90 \), \( f(2) = 80 \), \( f(3) = 95 \), \( f(4) = 60 \).

**Notation of function**: Usually a function is expressed by the notation \( f(x) \), \( F(x) \), \( g(x) \) etc.

**Value of function**: If \( f(x) \) is a given function, then \( f(\alpha) \) indicates the value of that function when \( \alpha \) is replaced by \( x \). For example, if \( f(x) = x^3 - 8x + 9 \), then \( f(2) = 2^3 - 8\times 2 + 9 = 8 - 16 + 9 = 17 - 16 = 1 \)

In the definition of a function from \( x \) into \( y \), it is mentioned that for each \( x \in X \), there exists a unique element \( y \in Y \). But for several elements of \( X \), the
corresponding elements of $y$ may be the same, such as, for the function described by $f(x) = x^2$, $f(x)$ and $f(\bar{x})$ are the same. Again, it maybe that, for two different elements of $X$, the corresponding elements of $Y$ are always different, as in case of the function in example 3. Such functions are called one-one functions.

**Example 4.** If $f(x) = x^4 + 5x - 3$, find the value of $f(\bar{1})$,

**Solution:** 

$$f(x) = x^4 + 5x - 3$$

$$\therefore f(\bar{1}) = (\bar{1})^4 + 5(\bar{1}) - 3 = 1\bar{1} - 3 = 1\bar{1} - 8 = \bar{7}$$

**Example 5.** If $f(x) = 2x - 6$, then for what value of $x$, $f(x) = 0$?

**Solution:** 

$$f(x) = 0$$

or, $2x - 6 = 0$ or, $2x = 6$ 

$$\therefore x = 3$$

**Answer:** for $x = 3$, $f(x) = 0$

**Exercise 7.2**

1. If $f(x) = x^2 - 2x + 6$, find the values of $f(2)$, $f(\bar{3})$ and $f\left(\frac{1}{3}\right)$.

2. If $f(x) = x^3 - 5x + 6$, for what values of $x$, $f(x) = 0$?

3. If $f(x) = x^3 + kx^2 - 4x - 8$, then for what values of $k$, $f(\bar{2}) = 0$?

4. If $g(x) = \frac{3x + 4}{x - 5}$, then what is the value of $g(6)$?

5. If $f(x) = \frac{3x + 1}{3x - 1}$, then what is the value of $\frac{f(x) + 1}{f(x) - 1}$?

6. For $f(x) = \frac{1 + x^2 + x^4}{x^2}$, show that, $f\left(\frac{1}{x}\right) = f(x)$.

**Graph**

A graph is a picture representation of the values of variables related by an algebraic equation. Since the graphs are the picture representations of equations, they are of much importance to make ideas clear of equations.

Moreover, relation between algebra and geometry is established through graphs. French philosopher and mathematician Rene Descartes (1596 - 1650) played the pioneer role to establish the fundamental relation between algebra and geometry. With the help of two straight lines intersecting perpendicularly in a plane, he described the position of a point precisely and there by established the modern
trends of plane geometry. He designated two perpendicularly intersecting straight lines as axes and called the point of intersection of axes the origin.

**Perpendicular axes and co-ordinate:** In a certain plane two perpendicularly intersecting straight lines $XOX'$ and $YOY'$ are drawn. The horizontal line $XOX'$ and the vertical line $YOY'$ are called the $x$-axis and $y$-axis respectively. The point of intersection of the axis $O$ is called the origin. Signed numbers indicating the directed perpendicular distances of a point in the plane from the two axes are called the co-ordinates of that point.

From any point $P$ lying on the plane, the directed perpendicular distance $PM$ of the $y$-axis is called the $x$-co-ordinate or the abscissa of $P$ and the directed perpendicular distance $PN$ of the $x$-axis is called the $y$-co-ordinate or the ordinate of $P$. In fact, the position of every point lying on the plane of the axes can be precisely described by means of its co-ordinates. The point is expressed in short by the point $(x, y)$. The co-ordinates mentioned above are called Cartesian co-ordinates.

**Note:**

(i) co-ordinates of the origin $(0, 0)$

(ii) distance of the point $(x_1, y_1)$ from $y$-axis is $|x_1|$

(iii) distance of the point $(x_1, y_1)$ from $x$-axis is $|y_1|$

(iv) ordinate of every point on $x$-axis is zero.

(v) abscissa of every point on $y$-axis is zero.

**Sign-rule of co-ordinates:**

In a cartesian co-ordinate system, the axes $XOX'$ and $YOY'$ divide the whole plane into four parts $XOY$, $YOX$, $X'OY'$ and $Y'OX$. These are called first, second, third and fourth quadrant respectively. The abscissa of any point lying on the right side of $y$-axis is positive and the abscissa of any point of the left side of $y$-axis is negative. Again, the ordinate of any point lying above the $x$-axis is positive and the ordinate of any point below the $x$-axis is negative. Thus to find out the sign of co-ordinates of points lying on different quadrants. We have the
following rules:

(i) in first quadrant, both $x$ and $y$ are positive.
(ii) in second quadrant, $x$ is negative, $y$ is positive
(iii) in third quadrant, both $x$ and $y$ are negative.
(iv) in fourth quadrant, $x$ is positive, $y$ is negative.

**Graph paper**: Graphs are pictures showing the relation between the variables connected by means of equations and to draw such graphs, graph papers are needed. A kind of paper with small squares drawn on it is used to find out the position of points on plane. Drawing some horizontal and vertical lines at equal distances, the paper is divided into small squares. This kind of squared paper is called the graph paper. Length of the side of one or more small squares on the graph paper may be taken as the unit.

**Plotting points on graph paper**: On a graph paper we draw $x$-axis and $y$-axis by taking one horizontal line and one vertical line designated as $XOX'$ and $YOY'$ respectively. Then what number of length of the side of the small squares should be taken as unit is determined according to need and advantage. The position of the point is then ascertained depending upon the unit and on the basis of sign of abscissa and ordinate. The process will be clear from the example given below.

**Example 6**: Plot the point $A(2, 4)$, $B(-3, 2)$, $C(-5, -7)$, $D(5, -10)$ on the graph paper.

**Solution**: Co-ordinate axes $XOX'$ and $YOY'$ are drawn through the middle of the graph paper and length of one side of the small square is taken as unit. Abscissa and ordinate of the point $A(2, 4)$ are both positive, so the point $A$ will lie in the first
from the origin O, we move 2 units along OX, then from there, moving 4 units parallel to OY we get the point. Marking the point we write by its side the co-ordinates of the point (2, 4). Co-ordinates of the point B are (–3, 2). Abscissa of this point is negative and its ordinate is positive. The point B will lie in the second quadrant. From the origin we move 3 units along OX and from there moving 2 units parallel to OY, we get the position of the point. Marking the point we write by its side the co-ordinates of the points (–3, 2). Co-ordinates of the points C are (–5, –7). Here abscissa and ordinate are both negative, the point lies in the third quadrant. Moving 5 units along OX’ from the origin and then from there moving 7 units parallel to OY’, we get the point. Co-ordinates of the point D are (5, –10). Abscissa of this point is positive and its ordinates is negative. The point D lies in the fourth quadrant. Moving 5 units along OX from the origin and from there moving 10 units towards OY’, we get the point.

**To find the distance between two points**: Let P (x₁, y₁) and Q (x₂, y₂) be two points. The perpendiculars PN and QM are drawn on OX from the points P and Q. Again perpendicular QK is drawn on PN from Q and P, Q are joined. Now applying Pythagorean theorem to the right-angled triangle PQK, we get,

\[ PQ^2 = QK^2 + PK^2 \]

or, \[ PQ^2 = (ON \parallel OM)^2 + (PN \parallel KN)^2 \]

or, \[ PQ^2 = (ON \parallel OM)^2 + (PN \parallel QM)^2 \]

or, \[ PQ^2 = (x_1 \parallel x_2)^2 + (y_1 \parallel y_2)^2 \]

\[ PQ = \sqrt{(x_1 \parallel x_2)^2 + (y_1 \parallel y_2)^2} \]

\[ = \sqrt{(\text{difference of abscissa})^2 + (\text{difference of ordinates})^2} \]

i.e. rectilinear distance between two points (x₁, y₁) and (x₂, y₂) is \[ d = \sqrt{(x_1 \parallel x_2)^2 + (y_1 \parallel y_2)^2} \]
N. B. This formula is applicable irrespective of the position of the points P, Q. To find out the distance between two points, only positive square root is to be taken. To be noted, distance of a point P(x, y) from the origin is \( OP = \sqrt{x^2 + y^2} \)

**Equation of Straight Lines Parallel to the Axes:**
Set of points which are lying at equal perpendicular distance from X axis is a straight line parallel to X axis. In other words we can say that the set of points, whose directed perpendicular distance from X axis are fixed (positive or negative numbers or zero), is a straight line.

Let, AB to such a straight line every point of which is at a perpendicular distance of b units from the X axis. Every point P(x, y) on the straight line satisfies the condition y = b. So equation of the straight line parallel to X-axis is y = b. For negative value of b, straight line parallel to X-axis will be b units below the X-axis. When b = 0, straight line AB will coincide with the X axis. So equation of X-axis is y = 0. Similarly, the equation of a straight line parallel to the Y-axis is x = a. In particular, the equation of Y axis is x = 0. For example, the equation x = 5 represents a straight line which is parallel to the Y-axis and lines on the left side of Y-axis at a distance of 5 units and the equation y = 6 represents a straight line which is parallel to the X-axis and lines 6 units above the X-axis.

**General Equation of a Straight Line:** Any linear equation \( ax + by + c = 0 \) comprising two variables x, y always represents a straight line \( ax + by + c = 0 \) is called the standard equation of a straight line. Equations representing straight lines are shown by using the graphs.
Equation of a Circle with Centre \((p, q)\) and Radius \(r\):

Let \(C(p, q)\) be the centre of the circle, \(r\) be its radius and \(P(x, y)\) be any point on the circle.

Then \(CP = r\).

\[
\therefore \sqrt{(x - p)^2 + (y - q)^2} = r
\]
or,

\[
(x - p)^2 + (y - q)^2 = r^2
\]

This equation is applicable for any position of the point \(P(x, y)\) on the circumference. So it is the equation of the circle.

**Corollary:** If the centre is at the origin \((0, 0)\), the equation of the circle is \(x^2 + y^2 = r^2\).

**Example 7.** Express \(x^2 + y^2 - 6x - 8y - 39 = 0\) in the form \((x - p)^2 + (y - q)^2 = r^2\) and mention the nature of the graph.

**Solution:**

Given, \(x^2 + y^2 - 6x - 8y - 39 = 0\)
or, \((x - 3)^2 + (y - 4)^2 - 64 = 0\)
or, \((x - 3)^2 + (y - 4)^2 = 64\)
or, \((x - 3)^2 + (y - 4)^2 = 8^2\)

Therefore, the graph of the given equation is the circle, of which the coordinates of centre is \((3, 4)\) and the radius is 8 units.

**Graphs of Linear Equation**

**Example 8.** Sketch the graph of the equation \(x = 5\)

**Solution:** The equation \(x = 5\) can be written as \(x + 0 \cdot y = 5\). It is to be noted here that whatever be the value of \(y\), value of \(x\) will be always 5. So we may take such values of \(x, y\) which satisfy the equation in the following way.

<table>
<thead>
<tr>
<th>(x)</th>
<th>5</th>
<th>5</th>
<th>5</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>(y)</td>
<td>Đ 2</td>
<td>6</td>
<td>10</td>
<td>Đ 4</td>
</tr>
</tbody>
</table>
Plotting the points (5, \(\pm 2\)), (5, 6), (5, 10), (5, \(\pm 4\)) on the graph paper and then joining the points we get the graph. Here the graph is parallel to the Y-axis and lies at a distance on the positive side of X-axis. Therefore the straight line which is at a distance of 5 units from the origin on the right side and parallel to YOY' is the required graph.

**Example 9.** Sketch the graph of the equation \(2x - 7y + 12 = 0\)

**Solution:** \(2x - 7y + 12 = 0\)

or, \(-7y = -2x - 12\)

or, \(7y = 2x + 12\)

\[\therefore y = \frac{2x + 12}{7}\]

From this relation we find the co-ordinates of some points of the graph.

<table>
<thead>
<tr>
<th>x</th>
<th>1</th>
<th>8</th>
<th>15</th>
</tr>
</thead>
<tbody>
<tr>
<td>y</td>
<td>2</td>
<td>4</td>
<td>6</td>
</tr>
</tbody>
</table>

Taking the length of the side of a small square on the graph paper as unit, we plot the points (1, 2), (8, 4), (15, 6) on the graph paper. After joining the points, we extend it on both sides and we get a straight line. This straight line is the graph of the equation \(2x - 7y + 12 = 0\)

**N. B.:** Since the graph of any equation in the form of \(ax + by + c = 0\) is a straight line, it is sufficient to plot two points in order to draw the graphs. But in practice it is desirable to plot at least three points in order to have possibility (of detecting any mistakes in counting or plotting the points).
Graph of Quadratic Equation:

Example 10. Sketch the graph of the equation $x^2 + y^2 - 8x - 10y - 103 = 0$ on a graph paper.

Solution:
Given, $x^2 + y^2 - 8x - 10y - 103 = 0$

or, $x^2 - 8x + 16 + y^2 - 10y + 25 - 16 - 25 - 103 = 0$

or, $(x - 4)^2 + (y - 5)^2 - 144 = 0$

or, $(x - 4)^2 + (y - 5)^2 = 144$

∴ $(x - 4)^2 + (y - 5)^2 = 12^2$

Graph of the given equation is a circle of which co-ordinates of the centre is $(4, 5)$ and radius is 12. Taking the length of the side of a small square on the graph paper as unit, we plot the point $(4, 5)$ on the graph paper. Let the point be $C$. Now with $C$ as centre and taking a radius of 12 units, we draw a circle. The circle drawn is the graph of the given quadratic equation.

Exercise 7.3

1. Plot the points $(3, 1)$, $(0, -5)$, $(-3, 4)$, $(7, -9)$ on a graph paper.
2. Plotting three points $(1, 2)$, $(1, 1)$, $(11, 7)$ on a graph paper, show that the three points lie on a straight line.
3. Find the distance between the points $(4, 7)$ and $(1, 5)$.
4. Find the equation of a circle whose centre is $(4, 3)$ and radius is 10.
5. Draw the graphs of the following equations:

   (i) $y = 7$
   (ii) $x = 10$
   (iii) $x = 3 - 4y$

   (iv) $\frac{x}{2} + \frac{y}{3} = 1$
   (v) $\frac{x}{2} - \frac{y}{3} = 1$
   (vi) $4x + 3y = 12$
(vii) \( x + y = 10 \)  
(viii) \( 7x - 3y = 21 \)  
(ix) \( 2y - 2x = 7 \)

(x) \( y = \frac{1}{2}x + 5 \)  
(xi) \( 2x - 9y - 5 = 0 \)

(xii) \( 3x - 5y - 16 = 0 \)

6. Sketch the graph of the equation \( x^2 + y^2 - 64 = 0 \) on a graph paper.

7. Sketch the graph of the equation \((x - 3)^2 + (y + 5)^2 - 81 = 0\)

8. Sketch the graph of the equation \( x^2 + y^2 - 6x - 8y - 75 = 0 \)

9. Sketch the graph of the equation \( 4x + 5y = 20 \). Find the length of the segment of the graph cut by the axes.

**Variation**

**Direct Variation**: If two variables are so related that with the increase or decrease of one variable, the other variable always increases or decreases in the same ratio then we say that one variable directly varies as the other or that one variable is in direct variation with the other. The fixed ratio in direct variation is called its constant of variation. For example, if the height of a triangle is constant, the area of the region enclosed by triangle will directly varies as the base. Because with the increase or decrease of the base, the area of the triangular region will increase or decrease in the same ratio. When \( x \) directly varies as \( y \), we write, \( x \propto y \) and we read this as, \( x \) varies as \( y \).

**Note**: If \( A \propto B \), then \( A = kB \), where \( k \) is a constant, Conversely, if \( A = kB \), where \( k \) is constant, then \( A \propto B \).

**Method of Finding the Constant of Variation**

**Example 11**. If \( A \propto B \) and \( A = 20 \) when \( B = 5 \), find the constant of variation.

**Solution**: Since \( A \propto B \)

\[
A = kB \\
\therefore \quad 20 = k \cdot 5 \\
\therefore \quad k = \frac{20}{5} \\
\therefore \quad k = 4
\]
Inverse Variation: If two variables are so related that with the increase of one, the other always decreases in the same or when one decreases, the other increases in the same ratio, then we say that one variable varies inversely as the other or that one variable is in inverse variation with the other. Thus a variable $y$ varies inversely as a variable $x$, if $y$ directly varies as $\frac{1}{x}$, that is if $y = k \cdot \frac{1}{x}$, where $k$ is a constant. Therefore, $x$ and $y$ varies inversely if and only if $xy = \text{constant}$. In any rectangular region with a fixed area, length and breadth vary inversely.

Example 12. $y \propto x$ and $y = 5$ when $x = 15$, find a relation between $x$ and $y$.

Solution: $y \propto x \ [\text{given}]$

$\therefore y = kx; \text{ where } k \text{ is a constant } \cdots \cdots (i)$

$\therefore \text{ Here, } 5 = 15k \ [\text{putting the given values of } x \text{ and } y]$  

$\therefore k = \frac{5}{15} = \frac{1}{3}$

Now in equation (i) putting $k = \frac{1}{3}$ we get, $y = \frac{1}{3}x$

$\therefore x = 3y$

Example 13. If $x \propto y$, prove that $x^2 - y^2 \propto xy$.

Solution: $x \propto y \ [\text{given}]$

$\therefore x = ky \ \text{[where } k \text{ is a constant ]} \cdots (i)$

Multiplying both sides of the equation (i) by $x$ we get,

$x^2 = kxy \ \text{.......................................................... (ii)}$

Again, multiplying both sides of the equation (i) by $\frac{y}{k}$ we get,

$y^2 = \frac{xy}{k} \ \text{.......................................................... (iii)}$

Now, subtracting equation (iii) from equation (ii) we get,

$x^2 - y^2 = xy \ (k - 1 \cdot \frac{1}{k}); \ \text{here } k \cdot \frac{1}{k} \text{ is a constant}.$

$\therefore x^2 - y^2 \propto xy$

Example 14. A solid sphere is formed by melting two solid spheres made of gold and having radii $r_1$ and $r_2$. What is the radius of the new sphere? Given that the volume of a sphere directly varies as the cube of its radius.

Solution: Let $v_1$ and $v_2$ be the volume of the spheres. Since the volume of a sphere directly varies as the cube of its radius, $v_1 = kr_1^3$ and $v_2 = kr_2^3$, where $k$ is a constant.
Let radius of the new sphere be $r$, where volume is $v_1 + v_2$

$$v_1 + v_2 = kr^3$$

or, $kr_1^3 + kr_2^3 = kr^3$ [putting the values of $v_1$ and $v_2$]

or, $k(r_1^3 + r_2^3) = kr^3$

or, $r^3 = r_1^3 + r_2^3$

$$r = \sqrt[3]{r_1^3 + r_2^3}$$

---

**Exercise 7.4**

1. $y \propto x$ and $y = 10$ when $x = 25$; when $x = 35$, find the values of $y$.

2. The square of $x$ directly varies as the cube of $y$ and $x = 2$, when $y = 3$. Express the relation between $x$ and $y$ as an equation.

3. If $a + b \propto a \cdot b$, show that, $a^2 + b^2 \propto ab$.

4. If $x \propto y$ and $y \propto z$, prove that $x^2 + y^2 + z^2 \propto yz + zx + xy$.

5. If $a \propto b$ and $b \propto c$, show that $(a^2 + b^2)^{\frac{3}{2}} \propto c^3$.

6. If $(r + s) \propto \left(t + \frac{1}{t}\right)$ and $(r \cdot D \cdot s) \propto \left(t \cdot D \cdot \frac{1}{t}\right)$ then express the relation between $r$ and $t$ as an equation, when it is given that $r = 3$, $s = 1$ when $t = 2$.

7. Given that the intensity of light at any point varies inversely as the square of the distance of that point from the source of light. A book placed at a distance of 6 metres from a table lamp, gets a certain amount of light from the lamp. How far the book is to be shifted from the table lamp to get half of that amount of light?

8. Distance traversed by an object falling freely from rest directly varies as the square of the time of its falling. If in 5 seconds, an object falls through 122\(\text{m}\)5 metres, then how far the object will fall in the sixth seconds?
Multiple Choice Questions (MCQ) :

1. If R is a relation from set A to set B, then which one of the following is correct?
   A. $R \subseteq A \times B$  
   B. $R \subseteq A$  
   C. $R \subseteq B$  
   D. $(A \times B) \subseteq R$

2. Given that,
   i. If $f(x) = 2x - 6$, then $f(3) = 0$
   ii. if $x \propto y$, $y \propto z$, then $x \propto z$
   iii. If $y = x^3 - 3x + 6$, then $x$ is called a function of $y$.

Which of the above statements are correct?
   A. i and ii  
   B. ii and iii  
   C. i and iii  
   D. i, ii and iii

Given that : $f(x) = x^2 + x - 12$

Answer questions no. (3 Ñ 5) based on the above information :

3. Which one of the following is the correct value of $f(3)$ ?
   A. $-3$  
   B. $0$  
   C. $4$  
   D. $12$

4. For which values of $x$, $f(x) = 0$?
   A. $3, -4$  
   B. $-3, 4$  
   C. $3, -12$  
   D. $-4, 12$

5. Which one of the followings is the subset of function?
   A. $\{ (0, -12), (3, 0), (4, 0) \}$  
   B. $\{ (3, 0), (4, 0), (5, 12) \}$  
   C. $\{ (4, 0), (4, 0), (5, 12) \}$  
   D. $\{ (0, -12), (4, 0), (3, 5) \}$
6. Given that,
   i. If $A = \pi r^2$, then $A$ is the function of $r$.
   ii. All the functions are relation,
   iii. If $x_1 \propto y_1$, $x_2 \propto y_2$, then $x_1y_1 \propto x_2y_2$

Which one of the following is correct based on the above statements?
   A. i and ii  
   B. i and iii 
   C. ii and iii  
   D. i, ii and iii

Creative Questions:

1. $f(x) = x^2 + y^2 - 6x - 8y - 75$
   A. Determine the value of $y$, if $f(x) = 0$
   B. Sketch the graph of the equations $f(x) = 0$
   C. Find the length of the segments of the graph $f(x) = 0$ cut by $x$ and $y$ axis.
Chapter VIII
System of Equations in Two Variables

Solution of equation in one variable have been discussed earlier. In this chapter we will consider system of equations in two variables.

\[ x - y = 4 \] is linear equation in two variables, as it contains two variables or unknown quantities \( x \) and \( y \) and it is clear that the equation may be satisfied by infinitely many pairs of values of the two unknown quantities e.g., \( x = 5, y = 1 \) or, \( x = 6, y = 2 \) or, \( x = 7, y = 3 \) or, \( x = 8, y = 4 \) or, \( x = \text{D}2, y = \text{D}6 \) etc. Now if the two linear equations \( x - y = 4 \) and \( x + y = 10 \) are considered simultaneously, then out of the infinitely many solutions of the equation \( x - y = 4 \), only solution \( x = 7, y = 3 \) also satisfies the second equation; in other words two equation \( x - y = 4, x + y = 10 \) are satisfied by \( x = 7, y = 3 \) only.

Given two equation in two unknowns if we are required to find the values of the unknown which satisfy the two given equations, the two equations are considered to form a system of equations in two variables and the pair of values of the unknown quantities which satisfy the system of equations are called the solution of the system of equations. For example, in the above system of equations, the only solution of the system is given by \( x = 7, y = 3 \). This solution may be expressed as ordered pair \((x, y) = (7, 3)\).

A system of equations in two variables may not have unique solution. For example, the system
\[
\begin{align*}
2x - 2y &= 8 \\
3x - 3y &= 12
\end{align*}
\]
has infinitely many solution \( x = 5, y = 1 \); \( x = 6, y = 2 \); \( x = 7, y = 3 \) etc. Here, though the two equations appear to be different, they are, in fact, equivalent to a single equation. If we multiply both sides of the first equation by \( \frac{3}{2} \), we get the second equations. Such system of equations are called dependent.

In general, if \( \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} \), then the system of equations
\[
\begin{align*}
a_1x + b_1y &= c_1 \\
a_2x + b_2y &= c_2
\end{align*}
\]
is dependent and such system of equations has infinitely many solutions. On the other hand some system of equation may not have any solution at all.
For example, system of equations
\[
x \neq y = 4 \\
3x \neq 3y = 10
\]
has no solution. Such system of equations are called in-consistent.

A system of equations
\[
a_1x + b_1y = c_1 ; \ a_2x + b_2y = c_2
\]
will be in-consistent if \( \frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2} \)

If the system of equations
\[
a_1x + b_1y = c_1 ; \ a_2x + b_2y = c_2
\]
has one or more solutions, then it is called consistent.

N. B. The system of equations \( a_1x + b_1y = c_1 ; \ a_2x + b_2y = c_2 \) will be consistent

(i) if \( \frac{a_1}{a_2} \neq \frac{b_1}{b_2} \) (or, \( a_1b_2 \neq a_2b_1 \)) [has a unique solution]

or, (ii) if \( \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} \) [has infinitely many solutions]

It will be in-consistent if \( \frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2} \) [has no solution ]

N. B. If \( c_1 = c_2 = 0 \), the system of equations will be always consistent. In that case, if \( \frac{a_1}{a_2} \neq \frac{b_1}{b_2} \), then the solution will be unique if \( \frac{a_1}{a_2} = \frac{b_1}{b_2} \), then the system will have infinitely many solutions.

**Example 1.** Explain whether the following system of equations are consistent and find the number of solutions,

(i) \( 4x + 3y = 7 \) \( 8x + 6y = 14 \)

(ii) \( 4x + 3y = 7 \) \( 8x + 6y = 9 \)

(iii) \( 4x + 3y = 7 \) \( 8x \neq 6y = 2 \)

**Solution:** In (i) \( \frac{4}{8} = \frac{3}{6} = \frac{7}{14} \)

\[ \therefore \text{the system of equations are consistent and has infinitely solutions.} \]

In (ii) \( \frac{4}{8} = \frac{3}{6} \neq \frac{7}{9} \)

\[ \therefore \text{the system of equations is in-consistent. It has no solution [the number of solution is zero].} \]
In (iii) $\frac{4}{8} \neq \frac{3}{6}$

∴ the system of equations is consistent and the solution is unique.

**Example 2.** Find which of the following system of equations has unique solution, no solution, infinitely many solutions.

(i) $5x + 2y = 16$  
(ii) $5x + 2y = 16$  
(iii) $5x + 2y = 16$

$7x - 4y = 2$  
$3x + 6y = 2$  
$\frac{15}{2}x + 3y = 24$

(iv) $5x + 2y = 0$  
(v) $5x + 2y = 0$

$10x + 4y = 0$  
$5x - 2y = 0$

**Solution:**

In (i) $\frac{a_1}{a_2} = \frac{5}{7}$, $\frac{b_1}{b_2} = \frac{2}{4} = \frac{1}{2}$  
∴ $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$

Hence the system has a unique solution.

In (ii) $\frac{a_1}{a_2} = \frac{5}{3}$, $\frac{b_1}{b_2} = \frac{2}{6} = \frac{1}{3}$, $\frac{c_1}{c_2} = \frac{16}{2} = 8$  
∴ $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$

Hence the system has no solution.

In (iii) $\frac{a_1}{a_2} = \frac{5}{15} = \frac{5}{3}$, $\frac{b_1}{b_2} = \frac{2}{3}$, $\frac{c_1}{c_2} = \frac{16}{24} = \frac{2}{3}$  
∴ $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

Hence the system has infinitely many solutions.

In (iv) $c_1 = 0$, $c_2 = 0$ and $\frac{a_1}{a_2} = \frac{1}{2}$, $\frac{b_1}{b_2} = \frac{1}{2}$  
∴ $\frac{a_1}{a_2} = \frac{b_1}{b_2}$

Hence the system has infinitely many solutions.

In (v) $c_1 = 0$, $c_2 = 0$ and $\frac{a_1}{a_2} = \frac{5}{15} = \frac{1}{3}$, $\frac{b_1}{b_2} = \frac{2}{6} = \frac{1}{3}$  
∴ $\frac{a_1}{a_2} = \frac{b_1}{b_2}$

Hence the system has a unique solution.
Exercise 8.1

1. Explain whether the following system of equations are consistent and find the number of solutions.
   (i) $3x - 4y = 10$  
   $6x - 8y = 18$
   (ii) $3x - 4y = 10$  
   $6x - 8y = 20$
   (iii) $3x - 4y = 10$  
   $6x + 5y = 46$

2. Mention which of the following system of equations has unique solution, no solution, many solutions.
   (i) \( \frac{1}{2}x + y = 1 \)  
   \( x + 2y = 2 \)  
   (ii) \( \frac{1}{2}x - y = 0 \)  
   \( x + 2y = 0 \)  
   (iii) \( \frac{1}{2}x + y = 1 \)  
   \( x + 2y = 1 \)  
   (iv) \( \frac{1}{2}x + y = 0 \)  
   \( x + 2y = 0 \)  
   (v) \( \frac{1}{2}x + y = 1 \)  
   \( x + y = 5 \)

Now we shall consider the system of equations in two variables which are independent and consistent. Such system of equations have always unique solution. Four methods for finding the solution will be discussed here:
(1) Method of Substitution, (2) Method of Elimination, (3) Determinant method and (4) Graphical method.

Method of Substitution

In this method from any one of the two given equations the value of one unknown is expressed in terms of the other and the value thus obtained is substituted in the equations.

Example 3. Solve by the method of substitution:
   \[ 4x + y = 2 \]  \[ 2x + 3y = 4 \]

Solution: We write any one of the two equations in the form \( y = ax + b \).

From equation (i) we get,
   \[ y = 2 - 4x \]

Substitution \( 2 - 4x \) for \( y \) in equation (ii) we get,
   \[ 2x + 3(2 - 4x) = 4 \]
   \[ 2x + 6 - 12x = 4 \]
   \[ -10x = -2 \]
   \[ x = \frac{1}{5} \]

Substitute \( x = \frac{1}{5} \) in equation (i) we get,
   \[ y = 2 - 4 \left( \frac{1}{5} \right) \]
   \[ y = \frac{6}{5} \]

Hence, the solution is \( (x, y) = \left( \frac{1}{5}, \frac{6}{5} \right) \).
or, \[ 10x = 10 \]
\[ \therefore \quad x = \frac{10}{10} = 1 \]

putting this value of \( x \) in equation (iii) we get,

\[ y = 2 \times 1 = 2 \]

Both equations \( 4x + y = 2 \) and \( 2x + 3y = 2 \) are satisfied by the values \( x = 1 \) and \( y = 2 \).

Therefore, the required solution is \((x, y) = (1, 2)\).

A pair of numbers which satisfies the system of equations is called a solution of the system of equations. It may be noted that the value of \( y \) may also be found by putting the value of \( x \) in equation (ii). By placing the values of \( x \) and \( y \) in the two given equations we see that the system of equations is satisfied by these values. Therefore, the solution is correct.

**N. B.** The student should always verify the correctness of the solution of the equations.

**Example 4.** Solve by the method of substitution.

\[ \frac{3 + x}{5} + \frac{y}{3} = 2 \]

\[ \frac{2(x + 1)}{3} \times \frac{y}{4} = 1 \]

**Solution:** We first make the two equations free from fraction. Multiplying both sides of the 1st equation by 15 we get,

\[ 3(x + 3) + 5(y - 2) = 30 \]

or, \[ 3x + 9 + 5y = 10 \quad \text{or,} \quad 3x + 5y = 31 \]

Multiplying both sides of the 2nd equation by 12 we get,

\[ 8(x + 1) \times 3(y - 1) = 12 \quad \text{or,} \quad 8x = 3y = 1 \]

From equation (i), by transferring we get, \( 3x = 31 \)

or, \[ x = \frac{31 \times 5y}{3} \]

**Example 4.** Solve by the method of substitution.
In equation (ii) substituting \( \frac{31}{3} 5y \) for \( x \) we get,

\[
8\left(\frac{31}{3} 5y\right) \cdot 3y = 1 \quad \text{or,} \quad 8\left(\frac{31}{3} 5y\right) \cdot 9y = 3
\]

or, \( 248 \cdot 40y \cdot 9y = 3 \quad \text{or,} \quad 49y = 425 \quad \therefore \quad y = \frac{425}{49} = 5
\]

Now, putting the value of \( y \) in equation (iii) we get, \( x = \frac{31}{3} \cdot 5.5 \)

\[
= \frac{6}{3} = 2
\]

\( \therefore \) the required solution is \( (x, y) = (2, 5) \).

**Exercise 8.2**

Solve the following system of equations by the substitution method

1. \( 2x + y = 8 \quad 3x \cdot 3y = 5 \)
2. \( 7x \cdot 3y = 31 \quad 9x \cdot 5y = 41 \quad 7x + 4y = 15 \)
3. \( \frac{1}{2}x + \frac{1}{3}y = 3 \quad \frac{x}{2} + \frac{y}{3} = 1 \quad \frac{2}{x} + \frac{3}{y} = 2 \quad \frac{x}{3} + \frac{y}{2} = 1 \quad \frac{5}{x} + \frac{10}{y} = \frac{5}{6} \)
4. \( x + 5y = 36 \quad a (x + y) = b (x \cdot 3y) = 2ab \quad \frac{x + y}{x \cdot 3y} = \frac{5}{3} \)
5. \( x \cdot 3y = 2a \quad \frac{x}{a} + \frac{y}{b} = 2 \quad a (x + y) = a \cdot 3y = a + y \cdot 2x \)
6. \( ax + by = a^2 + b^2 \quad ax + by = a^2 + b^2 \)

**Method of Elimination**

In this method, the equations are multiplied, if necessary, by two numbers such that the absolute value of the coefficients of one unknown in equations so formed are equal. To avoid multiplication by large number, generally the equations are multiplied by numbers so that the product is the L.C. M. of the two coefficients of the same variable. Then by adding or subtracting the two latter equations we get an equations, where only one unknown is present. As one unknown quantity is eliminated in this mehtod, it is called the method of elimination.
Example 5. Solve by the method of elimination:

\[ \begin{align*}
4x + y &= 2 \quad \text{(i)} \\
2x + 3y &= 4 \quad \text{(ii)}
\end{align*} \]

Solution: Multiplying equation (i) by 3 we get, \(12x + 3y = 6\) ............... (iii)

Subtracting equation (ii) from equation (iii) we get,

\[10x = 6 - 4 = 6 + 4 = 10\]

\[\therefore x = \frac{10}{10} = 1\]

Now in equation (i) putting \(x = 1\) we get, \(4 + y = 2\) \(\therefore y = 2 - 4 = -2\)

\[\therefore \text{the required solution is } (x, y) = (1, -2).\]

N. B. The solution can also be determined by multiplying equation (ii) by 2 and then subtracting the equation so formed from equation (i).

Example 6. Find the solution (assuming \(b \neq 0\)):

\[\begin{align*}
ax + by &= a^2 \\
bx - ay &= ab
\end{align*}\]

Solution: Given, \(ax + by = a^2\) .........................(i)

\[bx - ay = ab \quad \text{..........................(ii)}\]

Multiplying equation (i) and (ii) by \(a\) and \(b\) respectively we get,

\[\begin{align*}
a^2x + aby &= a^3 \\
b^2x - aby &= ab^2
\end{align*}\]

Adding equation (iii) and (iv) we get, \(a^2x + b^2x = a^3 + ab^2\)

or, \((a^2 + b^2)x = a(a^2 + b^2)\)

\[\therefore x = \frac{a(a^2 + b^2)}{a^2 + b^2} = a\]

Putting the value of \(x = a\) in equation (i) we get, \(a^2 + by = a^2\)

or, \(by = a^2 - a^2\) or, \(by = 0\), or, \(y = \frac{0}{b} = 0\)

\[\therefore \text{the required solution is } (x, y) = (a, 0).\]

N. B. As \(b \neq 0\), \(b^2 > 0\); therefore \(a^2 + b^2 < 0\), if \(b = 0\) and \(a \neq 0\), then we get the same solution \(x = a, y = 0\).
Example 7. Find the solution:
\[
\begin{align*}
\frac{x}{2} + \frac{y}{3} &= 1 \\
\frac{x}{3} + \frac{y}{2} &= 1
\end{align*}
\]
Solution: Given, \(\frac{x}{2} + \frac{y}{3} = 1\) ......................... (i)
\(\frac{x}{3} + \frac{y}{2} = 1\) ............................... (ii)

Multiplying equation (i) by 3 and equation (ii) by 2 we get,
\[
\begin{align*}
\frac{3x}{2} + y &= 3 \\
\frac{2x}{3} + y &= 2
\end{align*}
\]
Subtracting equation (iv) from equation (iii), we get,
\[
\frac{3}{2}x = 1 \quad \text{or,} \quad x = \frac{6}{5}
\]
Now, substituting \(x = \frac{6}{5}\) in equation (iv)
\[
\frac{2}{3} \cdot \frac{6}{5} + y = 2 \quad \text{or,} \quad y = 2 \cdot \frac{4}{5} = \frac{6}{5}
\]
∴ the required solution is \((x, y) = \left(\frac{6}{5}, \frac{6}{5}\right)\).

Example 8. Find the solution:
\[
\begin{align*}
81x + 62y &= 138 \\
62x + 81y &= 5
\end{align*}
\]
Solution: Given, \(81x + 62y = 138\) .................(i)
\(62x + 81y = 5\) ............................. (ii)

Adding equation (i) and (ii) we get,
\[
143x + 143y = 143 \quad \text{or,} \quad 143(x + y) = 143
\]
∴ \(x + y = 1\) .........................................................(iii)
∴ \(62(x + y) = 62\)
or, \(62x + 62y = 62\) .................................................(iv)

Subtracting equation (iv) from equation (i) we get, \(19x = 76\)
∴ \(x = \frac{76}{19} = 4\)
Putting $x = 4$ in equation (iii) we get,

$$4 + y = 1 \quad \therefore y = 1 \quad \therefore 4 = \therefore 3$$

$\therefore$ the required solution is $(x, y) = (4, \therefore 3)$.

**Exercise 8.3**

Solve by the method of elimination:

1. $2x + 3y = 7$
2. $6x - y = 1$
3. $7x - 3y = 31$
4. $\frac{x}{2} + \frac{y}{3} = 8$
5. $\frac{5}{x} + 3y = 8$
6. $\frac{x}{3} - \frac{2}{y} = 1$
7. $2x + \frac{3}{y} = 1$
8. $12x + 17y = 41$
9. $25x + 27y = 131$
10. $ax + by = ab$
11. $ax - by = ab$
12. $ax + by = c$

**Rule of Cross-Multiplication**

If $a_1x + b_1y + c_1z = 0$ and $a_2x + b_2y + c_2z = 0$, then

$$\frac{x}{b_1c_2 - b_2c_1} = \frac{y}{c_1a_2 - c_2a_1} = \frac{z}{a_1b_2 - a_2b_1}$$

**Proof:** Multiplying 1st equation by $c_2$ and 2nd equation by $c_1$ we get,

$$c_2a_1x + b_1c_2y + c_1c_2z = 0$$
and $$c_1a_2x + b_2c_1y + c_1c_2z = 0$$

Subtracting, $(c_2a_1 - b_1c_2) x + (b_1c_2 - b_2c_1) y = 0$

or, $D(\Delta a_1 c) x + D(\Delta b_2 c_1) y$

$\therefore \frac{x}{b_1c_2 - b_2c_1} = \frac{y}{c_1a_2 - c_2a_1}$

..................................................(1)
Again multiplying 1st equation by $b_2$ and 2nd equation by $b_1$ we get,

$$a_1b_2x + b_1b_2y + b_2c_1z = 0$$

and

$$a_2b_1x + b_1b_2y + b_1c_2z = 0$$

Subtracting, $(a_1b_2 - a_2b_1)x + (b_2c_1 - b_1c_2)z = 0$

or, $(a_1b_2 - a_2b_1)x = (b_1c_2 - b_2c_1)z$

\[\therefore \frac{x}{b_1c_2} = \frac{z}{a_1b_2 - a_2b_1} \quad \text{..............................}(2)\]

Therefore, from (1) and (2) we get,

$$\frac{x}{b_1c_2} = \frac{y}{c_1a_2 - c_2a_1} = \frac{z}{a_1b_2 - a_2b_1}$$

Written in proportional form, this rule is called the rule of cross-multiplication.

Putting $z = 1$ in the above two equations, we get

$$a_1x + b_1y + c_1 = 0$$

$$a_2x + b_2y + c_2 = 0$$

and the rule of cross-multiplication reduces to,

$$\frac{x}{b_1c_2} = \frac{y}{c_1a_2 - c_2a_1} = \frac{1}{a_1b_2 - a_2b_2}$$

Solving the above system of equations by finding the values of $x$ and $y$ from the above formula, is called the method of cross-multiplication. This is indeed an alternative form of the method of elimination.

**Example 9.** Solve and verify the solution:

$$2x + 3y + 7 = 0$$

$$3x + 2y + 8 = 0$$

**Solution:** By the rule of cross-multiplication,

$$\frac{x}{3 \times 8} = \frac{y}{7 \times 3} = \frac{1}{2 \times 2} = \frac{1}{3 \times 3}$$

or,

$$\frac{x}{24} = \frac{y}{21} = \frac{1}{4 \times 9}$$
or, \( \frac{x}{10} = \frac{y}{5} = \frac{1}{\mathcal{D} 5} \)

\( \therefore \) \( \frac{x}{10} = \mathcal{D} 2 \) and \( \frac{y}{5} = \mathcal{D} 1 \)

\( \therefore \) the required solution : \((x, y) = (\mathcal{D} 2, \mathcal{D} 1)\)

**Verification :**

On putting, \( x = \mathcal{D} 2 \) and \( y = \mathcal{D} 1 \)

the L. H. S. of the first equation

\( = 2 (\mathcal{D} 2) + 3 (\mathcal{D} 1) + 7 = \mathcal{D} 4 \mathcal{D} 3 + 7 = 0 \)

the L. H. S. of the second equation

\( = 3 (\mathcal{D} 2) + 2 (\mathcal{D} 1) + 8 = \mathcal{D} 6 \mathcal{D} 2 + 8 = 0 \)

\( \therefore \) the derived solution is correct.

**Example 10.** Solve with the help of the rule of cross-multiplication

\[ 3x - y - 7 = 0 \]
\[ 2x + y - 3 = 0 \]

**Solution :** By the rule of cross-multiplication,

\[
\frac{x}{(\mathcal{D} 1) \times (\mathcal{D} 3)} = \frac{y}{(\mathcal{D} 7) \times 2} = \frac{1}{(\mathcal{D} 3) \times 3} = \frac{3 \times 1}{\mathcal{D} 2 \times (\mathcal{D} 1)}
\]

or, \( \frac{x}{3 + 7} = \frac{y}{14 + 9} = \frac{1}{3 + 2} \)

or, \( \frac{x}{10} = \frac{y}{5} = \frac{1}{5} \)

\( \therefore \) \( x = \frac{10}{5} = 2 \)

\( \therefore \) \( y = \frac{5}{5} = \mathcal{D} 1 \)

\( \therefore \) the required solution : \((x, y) = (2, \mathcal{D} 1)\).
Exercise 8.4

By the method of cross-multiplication find the solution \( (x, y) \) and verify:

1. \( 2x + 3y + 5 = 0 \)
2. \( x + 2y = 7 \)
3. \( 3x - 5y + 9 = 0 \)
4. \( 4x + 7y + 6 = 0 \)
5. \( x + 2y = 7 \)
6. \( 5x - 3y = 0 \)
7. \( 3x - 5y + 9 = 0 \)
8. \( \frac{4x + 5y}{40} = x - y \)
9. \( y(3 + x) = x(6 + y) \)
10. \( (x + 7)(y - 3) + 7 = (y + 3)(x - 1) + 5 \)

Determinant Method

If \( a, b, c, d \) be any numbers, then
\[
\begin{vmatrix} a & b \\ c & d \end{vmatrix}
\]
is called determinant of second order and its value is \( ad - bc \).

i.e.
\[
\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.
\]

Using determinant, the solution of system of equations can easily be determined. Let us consider the system of equations.

\[
ax + by = p \\
\]
\[
\begin{align*}
\text{where, } ad - bc &\neq 0 \\
\text{Solution of this system of equations is } x &= \frac{pd - bq}{ad - bc}, \\
y &= \frac{aq - pc}{ad - bc}
\end{align*}
\]

Which can be obtained by the method of elimination or method of cross multiplication,

Noted that,
\[
\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.
\]
\[
\begin{vmatrix}
p & b \\
q & d \\
\end{vmatrix} = pd - bq.
\]
\[
\begin{vmatrix}
a & p \\
c & q \\
\end{vmatrix} = aq - pc.
\]

Hence, the above solution can be written in this form:

\[
x = \frac{\begin{vmatrix}
p & b \\
q & d \\
\end{vmatrix}}{\begin{vmatrix}
a & p \\
c & q \\
\end{vmatrix}}, \quad y = \frac{\begin{vmatrix}
a & b \\
c & d \\
\end{vmatrix}}{\begin{vmatrix}
a & b \\
c & d \\
\end{vmatrix}}
\]

The solution can be determined directly by using this method. This method is called determinant method.

**Remarks:** If \( ad - bc = 0 \), then the given system of equations is either inconsistent or dependent (i.e. equivalent to single equation). In the first case, there is no solution of system of equations and in the second case, there are innumerable solutions.

**Example 11.** Solve by the determinant method:

\[
6x - 2y = 6 \\
5x + y = 21
\]

**Solution:** Here the determinant formed with the co-efficient of \( x \) and \( y \) is,

\[
\begin{vmatrix}
6 & 5 \\
2 & 1 \\
\end{vmatrix} = 6.1 - 5.2 = 6 + 10 = 16
\]

\[
\therefore \ x = \frac{\begin{vmatrix}
6 & 5 \\
2 & 1 \\
\end{vmatrix}}{\begin{vmatrix}
6 & 5 \\
2 & 1 \\
\end{vmatrix}} = \frac{6.1 - 5.2}{16} = \frac{6 + 42}{16} = \frac{48}{16} = 3
\]

and \[
\begin{vmatrix}
6 & 6 \\
5 & 21 \\
\end{vmatrix} = 6.21 - 6.5 = 126 - 30 = 96
\]

\[
\therefore \ y = \frac{\begin{vmatrix}
6 & 6 \\
5 & 21 \\
\end{vmatrix}}{\begin{vmatrix}
6 & 5 \\
2 & 1 \\
\end{vmatrix}} = \frac{6.21 - 6.5}{16} = \frac{126 - 30}{16} = \frac{96}{16} = 6
\]

\[
\therefore \ \text{the required solution : } (x, y) = (3, 6).
\]
Example 12. Find the solution:

\[3x + 4 = 14\]
\[4x - 3y = 2\]

**Solution:** Here the determinant with the co-efficients of \(x\) and \(y\) is.

\[
\begin{vmatrix}
3 & 4 \\
4 & 3
\end{vmatrix}
= 3 (4) - 4 (3) = 12 - 12 = 0
\]

\[
\therefore x = \frac{14 \begin{vmatrix} 4 & 4 \\ 3 & 4 \end{vmatrix} - 2 \begin{vmatrix} 3 & 4 \\ 4 & 3 \end{vmatrix}}{\begin{vmatrix} 3 & 4 \\ 4 & 3 \end{vmatrix}}
= \frac{14 (4) - 2 (4) - 3 (4) + 3 (3)}{25}
= \frac{56 - 8 - 9}{25}
= \frac{39}{25}
\]

and \(y = \frac{3 \begin{vmatrix} 3 & 4 \\ 4 & 3 \end{vmatrix} - 4 \begin{vmatrix} 3 & 4 \\ 4 & 3 \end{vmatrix}}{\begin{vmatrix} 3 & 4 \\ 4 & 3 \end{vmatrix}}
= \frac{3 (4) - 4 (4)}{25}
= \frac{12 - 16}{25}
= \frac{-4}{25}
\]

\[
\therefore \text{the required solution: } (x, y) = (2, 2)
\]

Example 13. Solve (when \(a, b\) are not both 0):

\[bx - ay = 0\]
\[ax + by = a^2 + b^2\]

**Solution:** Determinant formed with the co-efficient of \(x\) and \(y\) is,

\[
\begin{vmatrix}
b & d \\
a & b
\end{vmatrix}
= b \cdot b - a \cdot (d)
= b^2 + a^2
\]

Hence, \(a^2 + b^2 > 0\) since \(a, b\) are not both zero.

\[
\therefore x = \frac{0 \begin{vmatrix} a & b \\ b & a \end{vmatrix} - a \begin{vmatrix} a & b \\ b & a \end{vmatrix}}{\begin{vmatrix} a & b \\ b & a \end{vmatrix}}
= \frac{0 (a^2 + b^2) - a (a^2 + b^2)}{a^2 + b^2}
= \frac{-a^3 - ab^2}{a^2 + b^2}
= a
\]

and \(y = \frac{b \begin{vmatrix} a & 0 \\ b & a \end{vmatrix} - a \begin{vmatrix} a & 0 \\ b & a \end{vmatrix}}{\begin{vmatrix} a & 0 \\ b & a \end{vmatrix}}
= \frac{b (a^2 + b^2) - a (a^2 + b^2)}{a^2 + b^2}
= \frac{b (a^2 + b^2)}{a^2 + b^2}
= b
\]

\[
\therefore \text{the required solution: } (x, y) = (a, b)
\]
Exercise 8.5

Find the solution \((x, y)\) by the determinant method:

1. \(4x - 2y = 2\)
   \(5x + y = 13\)

2. \(2x + 5y = 1\)
   \(x + 3y = 2\)

3. \(3x - 2y = 2\)
   \(5x - 3y = 5\)

4. \(x - y = 2a\)
   \(ax + by = a \cdot b\)

5. \(ax + by = a \cdot b\)
   \(bx \cdot ay = a + b\)

6. \(x + y = a + b\)
   \(ax \cdot by = a^2 \cdot b^2\)

7. \(\frac{x}{2} + \frac{y}{3} = 2\)
   \(2x + 3y = 13\)

8. \(\frac{x}{a} + \frac{y}{b} = a + b\)
   \(\frac{x}{a^2} + \frac{y}{b^2} = 2\)

9. \(bx + ay = \frac{2ab}{a^2 + b^2}\)

10. \(\frac{x}{a} + \frac{y}{b} = 2\)
    \(2bx + ay = 2ab\)

11. \(ax + by = a^2 + b^2\)
    \(2bx \cdot ay = ab\)

12. \(x + y = b\)
    \((b + c) x + (c + a) y = a \cdot b\)

13. \(\frac{x}{a} + \frac{y}{b} = 2\)
    \(ax \cdot by = a^2 \cdot b^2\)

14. \(\frac{x}{a} + \frac{y}{b} = a + b\)
    \(ax + by = a^3 + b^3\)

Graphical Method

By this method solutions are determined by drawing graphs. A system of simple equations in two variables consists of two linear equations. If we draw the graph of these two equations, we get two straight lines and the abscissa and the ordinate of their point of intersection give the solution of the given system of equations. The system of equations has no solution if the two straight lines are parallel.

Example 14. Solve graphically:

\(3x + y = 6\)
\(5x + 3y = 12\)

Solution: From the first equation we get, \(y = 6 - 3x\). We determine coordinates of some points on the graph of this equation:

<table>
<thead>
<tr>
<th></th>
<th>2</th>
<th>1</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>(x)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(y)</td>
<td>0</td>
<td>3</td>
<td>3</td>
</tr>
</tbody>
</table>
From the second equations we get,

$$3y = 12 - 5x$$

or, $$y = \frac{12 - 5x}{3}$$

We determine co-ordinates of some points on the graph of this equations;

<table>
<thead>
<tr>
<th>x</th>
<th>0</th>
<th>3</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>y</td>
<td>4</td>
<td>1</td>
<td>6</td>
</tr>
</tbody>
</table>

Taking twice the length of the side of a small square as unit, we plot the points (2, 0), (1, 3), (3, 3) on the graph of the first equation on a graph paper and then extend the line segment joining them on both sides. Again, using the same co-ordinate axes and the same unit, we plot the points (0, 4), (3, 1), (6, 6) on the graph of the second equation on the graph paper. We extend the line segment to both sides by joining them. It may be noted that both the graphs are straight lines. The two straight lines intersect each other at the point P.

Since the point P is common to both the straight lines, co-ordinate of this point will satisfy both the equations. From the graph, it is seen that the abscissa and the ordinate of P are 1.5 and 1.5 respectively.

∴ the required solution (x, y) = (1.5, 1.5)

**Example 15.** Solve by graphical method:

$$\frac{x}{2} + \frac{y}{3} = 4$$

$$\frac{x}{3} + \frac{y}{2} = \frac{13}{3}$$
Solution: From the first equation we get,

\[ 3x + 2y = 24 \]

or, \[ 2y = 24 - 3x \]

or, \[ y = \frac{24 - 3x}{2} \]

We determine co-ordinates of some points of this equation for graph:

<table>
<thead>
<tr>
<th>x</th>
<th>4</th>
<th>6</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>y</td>
<td>6</td>
<td>3</td>
<td>9</td>
</tr>
</tbody>
</table>

From second equation we get,

\[ 2x + 3y = 26 \]

or, \[ 3y = 26 - 2x \]

or, \[ y = \frac{26 - 2x}{3} \]

We determine the co-ordinates of some points of this equation:

<table>
<thead>
<tr>
<th>x</th>
<th>1</th>
<th>4</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>y</td>
<td>8</td>
<td>6</td>
<td>4</td>
</tr>
</tbody>
</table>

Taking the length of the side of a small square as unit, we plot on a graph paper the above mentioned points for the graph of the first equation and join them. The graph is a straight line, plotting the above mentioned points for the graph of the second equation (taking the same axes and the same unit as before on the same graph paper) we join them and extend it; this graph is also a straight line. The two straight lines intersect each other at the point P.

Since the point P lies on both the straight lines, the abscissa and the ordinate of the point P satisfy both the equations. It is seen from the graph that the abscissa and the ordinate of the point P are 4 and 6 respectively.

∴ the required solution \((x, y) = (4, 6)\).
Exercise 8.6

Find the solution (if there be any) by graphical method :

1. $3x - y = 5$  
   $3x - 2y = 4$
2. $2x + 5y = 7$  
   $8x + 11y = 19$
3. $3x + 4y = 1$  
   $3x + 2y = 4$
4. $x + y = 6$  
   $3x + 5y = 23$
5. $3x + 2y = 4$  
   $6x + 4y = 9$
6. $5x + 3y = 10$  
   $10x + 6y = 1$
7. $y - 2x + 3 = 0$  
   $2y + x - 5 = 0$

Use of Simultaneous Linear Equation

Many problems of daily life can be solved by using the notion of equations. On many occasions we are to find out the values of two unknown quantities. In such case we take x and y or any other two different symbols as the values of two unknown quantities. Then forming linearly independent, consistent equations from the condition of the problem, values of the unknown quantities x and y can be determined by solving the system of equations.

Example 16. If 1 is added to the numerator and the denominator of a function, it becomes $\frac{4}{5}$ and if 5 is subtracted from the numerator and the denominator, it becomes $\frac{1}{2}$. Find the fraction.

Solution : Let the fraction be, $\frac{x}{y}$

By the first condition, $\frac{x + 1}{y + 1} = \frac{4}{5}$ ..............................(i)

By the second condition, $\frac{x - 5}{y - 5} = \frac{1}{2}$ .............................. (ii)

From equation (i) we get,

$5(x + 1) = 4(y + 1)$

or, $5x + 5 = 4y + 4$

or, $5x - 4y = 1$ .............................................(iii)

from equation (ii) we get,

$2(x - 5) = y - 5$ ....................................................... (iv)
Applying the determinant method to equation (iii) and (iv) we get,

\[
\begin{align*}
    x &= \frac{\begin{vmatrix} D1 & D4 \\ 5 & 1 \\ 5 & 2 \end{vmatrix}}{\begin{vmatrix} D4 \\ 1 \end{vmatrix}} = \frac{1 + 20}{5 + 8} = \frac{21}{3} = 7 \\
    y &= \frac{\begin{vmatrix} 5 & D1 \\ 2 & 5 \\ 5 & 4 \end{vmatrix}}{\begin{vmatrix} D4 \\ 1 \end{vmatrix}} = \frac{25 + 2}{5 + 8} = \frac{27}{3} = 9
\end{align*}
\]

∴ The required fraction = \(\frac{7}{9}\)

N. B. System of equations, thus formed as above, may be solved by any method.

**Example 17:** In a number of two digits, the digit in the ones place is one more than 3 times the digit in the tens place. The number formed by interchanging the digits is eight times the sum of the digits. What is the number?

**Solution:** Let the tens place be \(x\) and that in the ones place be \(y\)

∴ the number = \(10x + y\)

From the first condition, \(y = 3x + 1\) ....................................(i)

Interchanging the digits, the number so formed is \(10y + x\). From the second condition, \(10y + x = 8(x + y)\),

∴ from (i), we get, \(10(3x + 1) + x = 8(x + 3x + 1)\)

or, \(31x + 10 = 32x + 8\) or, \(31x \equiv 32x = 8 \equiv 10\)

or, \(x \equiv 2\)

∴ \(x = 2\)

From equation (i) we get, \(y = 3x + 1 = 3(2) + 1 = 7\)

∴ the number is \(10x + y = 10(2) + 7 = 27\)

**Alternative Method:**

Suppose, the digit in the tens place = \(x\)

∴ the digit in the ones place = \(3x + 1\)

∴ the number is \(10x + (3x + 1) = 13x + 1\)
Again interchanging the place of the digits, the number becomes
\[10(3x + 1) + x = 31x + 10\]
From the second condition, \(31x + 10 = 8(x + 3x + 1)\)
or, \(31x + 10 = 32x + 8\)
or, \(x = 2\)
∴ The number = \(13x + 1 = 13.2 + 1 = 27\)

**Example 18.** The sum of the age of the father and the son is 50 years; when the age of the son will be equal to the present age of the father, the sum of their age will be 102 years. Find the present age of the father and the son.

**Solution:** Let the present age of the father be \(x\) years and the present age of the son be \(y\) years.
Therefore, by the first condition, \(x + y = 50\) .........................(i)
Difference of age between the father and the son is \(x - y\) years.
Therefore, after \(x - y\) years, the age of son will be \(x\) years and the age of father will be \(x + (x - y) = (2x - y)\) years.
By the second condition, \(x + (2x - y) = 102\)
or, \(3x - y = 102\) ..................................(ii)
Adding equations (i) and (ii) we get,
\[4x = 152\]
or, \(x = \frac{152}{4} = 38\)
∴ \(x = 38\)
Putting the value of \(x\) in equation (i) we get,
\(y = 50 - x = 50 - 38 = 12\)
∴ \(y = 12\)
Therefore, the present age of the father is 38 years and the present age of the son is 12 years.

**Example 19.** Investing taka \(x\) at 4% simple profit and taka \(y\) at 5% simple profit, a man earns a profit of taka 920 annually. If he had invested taka \(x\) at 5% simple profit and taka \(y\) at 4% simple profit, then his annual income would have been taka 880. Find the values of \(x\) and \(y\).

**Solution:** By the first condition, \(\frac{4x}{100} + \frac{5y}{100} = 920\)
or, \(4x + 5y = 92000\) ..................(i)
By the second condition, \( \frac{5x}{100} + \frac{4y}{100} = 880 \)
or, \( 5x + 4y = 88000 \) .........................(ii)
Adding equation (i) and (ii) we get,
\[ 9(x + y) = 180000 \]
or, \( x + y = 20000 \)
\[ \therefore 4x + 4y = 80000 \] ...........................(iii)
Subtracting equation (iii) from equation (i) we get, \( y = 12000 \)
Again, \( x + y = 20000 \)
\[ \therefore x = 20000 \] \( \therefore y = 20000 \) \( \therefore 12000 = 8000. \)
Answer: The man invested taka 8000 at 4% profit and taka 12000 at 5% profit.

Example 20. If the length of a rectangular field is increased by 3 metres and its breadth is reduced by 3 metres, then the area of the field is reduced by 18 square metres. Again if the length is increased by 3 metres and the breadth is increased by 3 metres, the area is increased by 60 square metres. Find the length and breadth of the field.

Solution: Let the length and breadth of the rectangular field be \( x \) metre and \( y \) metre respectively.
\[ \therefore \text{area of the field} = xy \text{ square metres}. \]
By the 1st condition, \( (x + 3) (y - 3) = xy - 18 \) ..................(i)
By the 2nd condition, \( (x + 3) (y + 3) = xy + 60 \) ...................(ii)
From equation (i) we get, \( 3y - 3x - 9 = -18 \)
or, \( 3(y - x) = -9 \)
or, \( y - x = -3 \) ............................(iii)
From equation (ii) we get, \( 3y + 3x + 9 = 60 \)
or, \( 3(y + x) = 51 \)
or, \( y + x = 17 \) ............................(iv)
Adding equation (iii) and (iv) we get, \( 2y = 14 \) \( \therefore y = 7 \)
Now putting the value of \( y \) in equation (iv) we get,
\[ 7 + x = 17 \] \( \therefore x = 17 \) \( - 7 = 10 \)
\[ \therefore \text{the length of the field is 10 metres and its breadth is 7 metres}. \]
Exercise 8.7

1. If 1 is subtracted from numerator of a fraction and 2 is added to the denominator, then the fraction becomes $\frac{1}{2}$ and if 7 and 2 are subtracted from the numerator and denominator respectively, it becomes $\frac{1}{3}$. Find the fraction.

2. If 2 is added to the numerator and the denominator of a fraction then the fraction becomes $\frac{7}{9}$. Again, if 3 is subtracted from the numerator and the denominator, the fraction becomes $\frac{1}{2}$; find the fraction.

3. Sum of the digits of a number consisting of two digits is 6. If the places of the digits are interchanged, the number so formed is three times the digit in tens place of the original number. What is the number?

4. One digit of a number consisting of two digits is 1 more than other. If the places of the digits are interchanged, the number becomes $\frac{5}{6}$ times the previous number. What is the number?

5. The difference of the digits of a number consisting of two digits is 4 and the number so formed by changing the places of the digits is 110. Find the number.

6. A number of three digits is three times the sum of its digits. If the number is multiplied by 3, the product is equal to the square of the sum of the digits. What is the number?

7. Eight years ago, the age of the father was eight times the age of the son. After ten years, the age of father will be twice the age of the son. What are their present age?

8. The present age of the father is five times the sum of the ages of his two sons. After ten years, the age of the father will be twice the sum of the ages of those two sons. What is the present age of the father?

9. The sum of the present ages of the father and the son is $y$ years and the difference of their present ages is 22 years. After 12 years, the age of father will be twice the age of the son. What is the value of $y$? What is the present age of the son?
10. Investing Tk. 4000 at \( x\% \) simple profit and Tk. 5000 at \( y\% \) simple profit. I get a profit of Tk. 320 yearly. But if I had invested Tk. 5000 at \( x\% \) simple profit and Tk. 4000 at \( y\% \) simple profit, then the annual profit would have been Tk. 310. Find the values of \( x \) and \( y \).

11. A boat goes 15 km per hour when rowed in favour of the current and it goes 5 km. per hour against the current. Find the speed of the current.

12. Rowing in favour of the current, a man reached a place in \( 2 \frac{1}{2} \) hours and came back in \( 3 \frac{3}{4} \) hours against the current. What is the ratio of the rowing speed and the current speed?

13. If the length of a rectangular field is reduced by 5 metres and the breadth is increased by 3 metres, then the area is reduced by 9 square metres. On the other hand if the length is increased by 3 metres and the breadth is increased by 2 metres, then the area is increased by 67 sq. metres. Find the length and the breadth of the rectangle.

14. If the length of a rectangular field is reduced by 5 metres and the breadth is increased by 3 metres, then the area remains unchanged. The area also remains unchanged if the length is increased by 5 metres and the breadth is reduced by 2 metres. Find the length and the breadth of the rectangle.

15. If in the triangle \( ABC \), \( \angle B = 6x \) degrees, \( \angle C = 5x \) degrees, \( \angle A = y \) degrees and \( 6 \angle A = 7 \angle B \), then find the values of \( x \) and \( y \).

16. In the cyclic quadrilateral \( ABCD \), \( \angle A = (4x + 3) \) degrees, \( \angle B = 2 (y - 1) \) degrees, \( \angle C = (2x + 17) \) degrees and \( \angle D = (5x + 2) \) degrees. Find the values of \( x \) and \( y \). [Hints : In a cyclic quadrilateral, the sum of the two opposite angles = 2 right angles.]

17. \( x \) labourers make a contract to finish a work in \( x \) days. But the work was finished in \( 2x \) days as \( y \) labourers were absent. Show that, \( x = 2y \).

18. A man in service receives a monthly salary. He gets a fixed increment (increase in salary) annually. If his monthly salary becomes Tk. 3500/- after 4 years and Tk. 4250/- after 10 years, what is his initial monthly salary and what is the annual increment?

19. In a chemistry laboratory a student observes that, amount of acid in a bottle is 20% of the solution and in another bottle amount of acid is 30% of the solution. What quantity of solution from each bottle should be mixed so that the amount of acid will be 27% in 100 millilitre (ml) solution?
**Quadratics Simultaneous Equations**

Method of solving systems of equations for one linear equation and one quadratic equation is shown here with the help of some examples.

Suppose, \( x + y = 5 \) ...................(i)
and \( x^2 + y^2 = 13 \) ...................(ii) are to be solved.

Putting the value of \( y = 5 - x \) from equation (i) in equation (ii) we get,
\[
x^2 + (5 - x)^2 = 13 \quad \text{(it is a quadratic equation in one variable)}
\]
or,
\[
x^2 + 25 - 10x + x^2 = 13
\]
or,
\[
2x^2 - 10x + 12 = 0
\]
or,
\[
2(x^2 - 5x + 6) = 0
\]
or,
\[
(x - 2)(x - 3) = 0
\]
\[
\therefore \quad x = 2 \quad \text{or,} \quad 3.
\]

Putting \( x = 2 \) in equation (i) we get, \( y = 3 \)

On the other hand, putting \( x = 3 \) in equation (i) we get, \( y = 2 \)

Therefore, two solutions of the given systems of equations are \((x, y) = (2, 3)\) and \((x, y) = (3, 2)\)

**Example 21.** Solve : \( x^2 + y^2 = 45 \)
\( xy = 18 \)

**Solution :**
\[
(x + y)^2 = x^2 + y^2 + 2xy
\]
\[
= 45 + 2.18 = 81 \quad [\therefore \quad xy = 18]
\]
\[
\therefore \quad x + y = \pm \sqrt{81} = \pm 9
\]

Again, \((x + y)^2 = x^2 + y^2 - 2xy = 45 - 2.18 = 9\)
\[
\therefore \quad x + y = \pm 3
\]

Suppose, \( x + y = 9 \) and \( x \neq y = 3 \)
Solving these two equations we get, \( x = 6, y = 3 \)

Again, taking \( x + y = 9 \) and \( x \neq y = 3 \) and solving we get, \( x = 3, y = 6 \)
Again, taking \( x + y = 9, x \neq y = 3 \) and solving we get, \( x = 3, y = 6 \).
Finally, taking \( x + y = 9 \) and \( x \neq y = 3 \) and solving we get, \( x = 6, y = 3 \).
\[
\therefore \quad \text{Required solution :} \ (x, y) = (6, 3), (3, 6), (6, 3)
\]

**N. B.** Here, each given equations, is of degree 2 and \( 2 \times 2 = 4 \) solutions are found.
Example 22. Solve : \( x + y = 2 \) and \( xy = 8 \)

Solution : \( x + y = 2 \) ...........................(i) \( \) and \( xy = 8 \) ...........................(ii)

From equation (i) we get, \( y = x - 2 \)

Putting \( y = x - 2 \) in equation (ii) we get,
\( x(x - 2) = 8 \) or, \( x^2 - 2x - 8 = 0 \)

or, \( (x - 4)(x + 2) = 0 \)

\( \therefore \) \( x = 4, -2 \)

Putting the values of \( x \) in equation (i) or (ii) we get, \( y = 2, -4 \) respectively.

\( \therefore \) Required solution \((x, y) = (4, 2), (-2, -4)\).

Alternative Method :
\( (x + y)^2 = (x - y)^2 + 4xy = 2^2 + 4 \cdot 8 = 36 \)

\( \therefore \) \( x + y = \pm 6 \) .............................. (iii)

When \( x + y = 6 \) we get,
\( (x, y) = (4, 2) \)

On the other hand, when \( x + y = -6 \) we get, \( (x, y) = (-2, -4) \)

\( \therefore \) Required solution \((x, y) = (4, 2), (-2, -4)\)

Example 23. Solve : \( 6x^2 + 7xy - 3y^2 = 90 \)
\( 2x + 3y = 18 \)

Solution : \( 6x^2 + 7xy - 3y^2 = 90 \) .............................. (i)
\( 2x + 3y = 18 \) .............................. (ii)

\( \therefore \) Left side of (i) \( = 6x^2 + 7xy - 3y^2 \)
\( = 6x^2 + 2xy + 3y(2x + 3y) \)
\( = 2x(3x + 3y) + 3y(2x + 3y) \)
\( = (2x + 3y)(3x + 3y) = 18(3x + 3y) \) \( [ \because 2x + 3y = 18 ] \)

\( \therefore \) \( 18(3x + 3y) = 90 \)

or, \( 3x + 3y = 5 \) .............................. (iii)

or, \( 9x + 3y = 15 \) .............................. (iv)

\( \therefore \) Adding (ii) and (iv) we get, \( 11x = 33 \) \( \therefore x = 3 \)

Then from equation (iii) we get, \( y = 3x - 5 = 3.3 - 5 = 4 \)

\( \therefore \) Required solution \((x, y) = (3, 4)\)

N. B. : Given system of equations is equivalent to the linear system of equations
\( 3x + y = 5; \ 2x + 3y = 18 \)

So, only one solution is obtained.
Exercise 8.8

Solve :
1. $x^2 + y^2 = 25$
   $x - 2y = 10$
2. $2x^2 + y^2 = 3$
   $x + y = 2$
3. $x^2 + y^2 = 61$
   $xy = \mathcal{D} 30$
4. $x^2 + y^2 = 85$
   $xy = 42$
5. $2x + y = 7$
   $xy = 3$
6. $x^2 - y^2 = 45$
   $x + y = 5$
7. $x^2 - y^2 = 99$
   $x - y = 9$
8. $\sqrt{\frac{x}{y}} + \sqrt{\frac{y}{x}} = \frac{5}{2}$
9. $2x + y = 7$
   $x^2 - xy = 6$
10. $x^2 - xy + y^2 = 21$
   $x + y = 3$
11. $x^2 + xy + y^2 = 3$
   $x^2 - xy + y^2 = 7$
12. $\frac{1}{x} + \frac{1}{y} = 7$

Application of Quadratic Simultaneous Equation

Example 24. The sum of the areas of two square regions is 650 square metres. If the area of the rectangular region formed by the two sides of the two squares is 323 square metres, what are lengths of the sides of the two squares?

Solution : Suppose, the length of the side of one square is $x$ metres and that of the other square is $y$ metres.

By the question, $x^2 + y^2 = 650$ .........................(i)
$xy = 323$.................................(ii)

$\therefore (x + y)^2 = x^2 + y^2 + 2xy = 650 + 646 = 1296$
$\therefore x + y = \pm \sqrt{1296} = \pm 36$

Now, $(x - y)^2 = x^2 + y^2 - 2xy = 650 - 646 = 4$
$\therefore x \pm y = \pm 2$

Since the length is positive, the value of $x + y$ must be positive,

$x + y = 36$ ..............(iii); $x \pm y = \pm 2$ ..............(iv)
Adding (iii) and (iv), \(2x = 36 \pm 2\)

or, \(x = \frac{36 \pm 2}{2} = 18 \pm 1 = 19\) or, 17

From equation (iii), \(y = 36 \pm x = 17\) or, 19.

\[\therefore\] The length of the side of one square is 19 metres and the length of the side of the other square is 17 metres.

**Example 25.** Twice the breadth of a rectangle is 10 metres more than its length. If the area of the region enclosed by the rectangle is 600 square metres, find its length.

**Solution :** Suppose that the length of the rectangle = \(x\) metres and breadth of the rectangle = \(y\) metres.

From the question, \(2y - x = 10\) \(........................(i)\)

\(xy = 600\) \(...........................(ii)\)

from equation (i), \(2y = 10 + x\) or, \(y = \frac{10 + x}{2}\)

Putting the value of \(y\) in equation (ii) we get,

\[x\left(\frac{10 + x}{2}\right) = 600\quad or,\quad \frac{10x + x^2}{2} = 600\]

or, \(x^2 + 10x = 1200\)

or, \(x^2 + 10x - 1200 = 0\)

or, \((x + 40)(x - 30) = 0\)

Therefore, \(x + 40 = 0,\) or, \(x = 30\)

that is, \(x = 40\) or, \(x = 30\)

But length can not be negative,

\[\therefore\] \(x = 30\)

Hence, the length of the rectangle = 30 metres.

**Example 26 :** If a number of two digits be divided by the product of its digits., the quotient is 3. When 18 is added to the number, the digits of the number change their places. Find the number.

**Solution :** Suppose, tens places digit = \(x\) and ones places digit = \(y\)

\[\therefore\] the number = \(10x + y\).
From the 1st condition, \( \frac{10x + y}{xy} = 3 \)

or, \( 10x + y = 3xy \) ...............................(i)

From the 2nd condition, \( 10x + y + 18 = 10y + x \)

or, \( 9x - 9y + 18 = 0 \)

or, \( x - y + 2 = 0 \)

or, \( y = x + 2 \) .................................(ii)

In equation (i) putting \( y = x + 2 \) we get,

\[ 10x + x + 2 = 3x(x + 2) \]

or, \( 11x + 2 = 3x^2 + 6x \)

or, \( 3x^2 - 5x - 2 = 0 \)

or, \( 3x^2 - 6x + x - 2 = 0 \)

or, \( 3x(x - 2) + 1(x - 2) = 0 \)

or, \( (x - 2)(3x + 1) = 0 \)

Therefore, \( x - 2 = 0 \) or, \( 3x + 1 = 0 \)

\[ \therefore x = 2 \] or, \( 3x = -1 \) or, \( x = -\frac{1}{3} \)

But the digit or a number can not be negative or fraction.

Therefore, \( x = 2 \) and \( y = x + 2 = 2 + 2 = 4 \)

\[ \therefore \text{the required number} = 24. \]

**Exercise 8.9**

1. The sum of the areas of two square regions is 481 square metres; if the area of the rectangle formed by the two sides of the two squares is 240 square metres, what is the length of side of each of the squares?

2. The sum of squares of two positive numbers is 250 ; the product of the numbers is 117 ; find the two numbers.

3. The sum of squares of two numbers is 13 and the product of the numbers is 6. Find the difference of the squares of the two numbers.

4. The sum of squares of two numbers is 181 and the product of the numbers is 90. Find the difference of the squares of the two numbers.
5. The area enclosed by a rectangle is 24 square metres. The length and breadth of another rectangle are respectively 4 metres and 1 metre more than the length and breadth of the first rectangle and the area enclosed by the later rectangle is 50 square metres. Find the length and breadth of the first rectangle.

6. Twice the breadth of a rectangle is 23 metres more than its length. If the area enclosed by the rectangle is 600 square metres, find the length and breadth of the rectangle.

7. The perimeter of a rectangle is 8 metres more than the sum of its diagonals. If the area enclosed by the figure is 48 square metres; find the length and breadth.

8. If a number of two digits be divided by the product of its digits, the quotient is 2. When 27 is added to the number, digits in the number change their places. Find the number.

9. The perimeter of a rectangular garden is 56 metres and one diagonal is 20 metres. What is the length of the side of the square which encloses an area equal to the area of that garden?

10. The area of a rectangular field is 300 square metres and its semiperimeter is 10 metres more than a diagonal. Find the length and breadth of the rectangular field.
Multiple Choice Questions (MCQ) :

1. Based on the following which conditions, the system of equation $a_1x + b_1y = c_1$ and $a_2x + b_2y = c_2$ will be consistent?
   
   A. $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$
   
   B. $\frac{a_1}{a_2} = \frac{b_1}{b_2}$
   
   C. $\frac{b_1}{b_2} = \frac{c_1}{c_2}$
   
   D. $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$

2. The solution of the system of equation $ax = 0$ and $a^2x + b^2y = b^3$ is $\bar{N}$
   
   A. $(a, b)$
   
   B. $(0, b^3)$
   
   C. $(0, b)$
   
   D. $(a^2, b^2)$

3. Mr. Arefin distributed $y$ number of mangoes among $x$ number of boys so that each of them got 6 mangoes and 6 more mangoes were remained. Which one of the following equations represents the above statement?
   
   A. $x = 6y + 6$
   
   B. $y = 6x + 6$
   
   C. $x = 6y - 6$
   
   D. $y = 6x - 6$

4. The difference and product of square of two positive integers are 3 and 2 respectively. Sum of their square is $\bar{N}$
   
   A. 1
   
   B. 2
   
   C. 3
   
   D. 5

5. Look at the following mathematical sentences $\bar{N}$
   
   i. The graph of the equation $\frac{x}{3} + \frac{y}{4} = 1$ passes through the point $(3, 0)$.
   
   ii. $\begin{vmatrix} a & x \\ b & y \end{vmatrix} = ax \not\equiv by$
   
   iii. $x^2 + y^2 = 9$ is a equation of a circle.

   Which one of the following is correct based on the above statements?
   
   A. i and ii
   
   B. i and iii
   
   C. ii and iii
   
   D. i, ii and iii

6. Look at the following mathematical sentences $\bar{N}$
   
   i. $x \not\equiv 3y = 4$ is a equation of a straight line.
   
   ii. $4x = 5$ is a equation of a line parallel to the x axis
   
   iii. The intersection point of $x = a, y = b$ is $(a, b)$

   Which one of the following is correct based on the above statements?
   
   A. i and ii
   
   B. i and iii
   
   C. ii and iii
   
   D. i, ii and iii
Anik and Ayesha have x and y numbers of oranges respectively. Anik has 2 more oranges than Ayesha.

Answer question no. (7 Ñ 9), based on the above information:

7. Which one of the following equations represents the above statements?
   A.  \( x + 2 = y \)  
   B.  \( x = y + 2 \)  
   C.  \( x + y = 2 \)  
   D.  \( x + y + 2 = 0 \)  

8. If Ayesha has one orange., then how many oranges both of them together have?
   A.  1  
   B.  2  
   C.  3  
   D.  4  

9. What is the total price of oranges both of them have if price of an orange is Tk. 5?
   A.  4  
   B.  5  
   C.  15  
   D.  20  

Creative Questions:

1. The length and breadth of a rectangular garden are x metre and y metre respectively, where the relation between length and breadth can be depicted as the following two equations: \( \frac{x}{7} + \frac{y}{3} = \frac{67}{7} \) and \( \frac{x}{5} - \frac{y}{4} = \frac{1}{2} \)
   A. Transform the given equations in the form of ax + by = c.
   B. Determine the length and breadth of the garden by solving the equations using the method of elimination.
   C. There is a road of 3 metre width inside around the garden. Find the number of stones of area 50 sq. cm requires for covering the road.

2. Given that:
   \( 3x + y = 5 \)
   \( 3x + 2y = 4 \)
   A. Explain whether the system of equation is consistent. Find the number of solutions.
   B. Find the solution of \((x, y)\) by the method of cross-multiplication.
   C. Find the solution of the system of equations by graphical method and verify the solutions obtained from question no. (B).
Chapter IX
Finite Series

If some numbers or expressions are arranged as first, second, third, .......... in succession, then we get a sequence. The first, second, third, ....... numbers or expressions occurring in the sequence so formed are respectively called its first term, second term, third term, ..... For example, 3, 5, 9, 14, 29, 43, ...... are respectively its first term, second term, third term etc. of the sequence formed by these numbers. In the above example it can not be ascertained whether the sequence has a last term or, even it if has one, what is it. On the other hand in the sequence 2, 4, 6, 8, ........... 40, the last term is 40. If terms of a certain sequence are connected in succession by '+ ' sign, we get a series. 3 + 7 + 11 + ... is a series and 3, 7, 11, ..... are respectively first, second, third term of the sequence. If the terms of a sequence are arranged according to some rule then the general term or the r-th term can be easily found, such as, in the sequence 2, 4, 6, ...... the r-th term is 2r.

Arithmetic Series :
2+4+6+... +20 is a series whose first term is 2, second term is 4, third term is 6.

Here, Second term Ð First term = 4 Ð 2 = 2.
Third term Ð Second term = 6 Ð 4 = 2

In this series difference of any term and its preceding term is always the same number.

Difference of two terms thus obtained is called the common difference. Common difference in the above mentioned series is 2. The number of terms of the series is fixed. It is a finite (or terminating) series. A series in which difference is obtained by subtracting any term from the next term is called an arithmetic series and this difference is called the common difference in the series. It is to be noted that, the common difference may be positive or negative.

\[ r\text{-th term } = 5 + (r - 1) \times 3 = 3r + 2 \]
**Formula**: In an arithmetic series, if the 1st term is \( a \) and the common difference is \( d \), then the \( r \)th term = \( a + (r - 1) \cdot d \)

**Example 1.** Which term of the series 5 + 8 + 11 + 14 + .... is 302?

**Solution**: It is an arithmetic series whose 1st term is \( a = 5 \), the common difference \( d = 8 - 5 = 3 \)

Suppose, the \( r \)-th term = 302.

∴ \( r \)-th term = \( a + (r - 1) \cdot d \)

∴ \( a + (r - 1) \cdot 3 = 302 \)

or, \( 5 + (r - 1) \cdot 3 = 302 \)

or, \( (r - 1) \cdot 3 = 302 - 5 = 297 \)

∴ \( r - 1 = \frac{297}{3} = 99 \)

or, \( r = 99 + 1 = 100 \)

∴ 100-th term of the given series = 302

**Sum of n Terms of an Arithmetic Series**

**Example 2.** What is the sum of 25 terms of the series 7 + 12 + 17 + .......?

**Solution**: This is an arithmetic series whose 1st term \( a = 7 \).

Common difference \( d = 12 - 7 = 5 \). Number of terms, \( r = 25 \)

∴ 25th term = \( a + (r - 1) \cdot d \)

= \( 7 + 24 \times 5 = 127 \)

Suppose, the sum of 25 terms = \( S \)

∴ \( S = 7 + 12 + 17 + ....... + 117 + 122 + 127 \)

Writing the terms in the reverse order, \( S = 127 + 122 + 117 + .... + 17 + 12 + 7 \)

Adding corresponding terms of the two series we get,

\[
2S = 134 + 134 + 134 + .... + 134 + 134 + 134 \\
= 134 \times 25 \quad [\text{since the number of terms = 25}] \\
\therefore S = \frac{134 \times 25}{2} = 67 \times 25 = 1675
\]

As in the above solution, we get the general formula stated below.

Suppose, in an arithmetic series with 1st term \( a \) and common difference \( d \), the sum of \( n \) terms is \( S \) and the last term of the series is \( p \). So we can write,

\[
S = a + (a + d) + (a + 2d) + ....... + (p - 2d) + (p \cdot d) + p \quad \ldots \ldots \ (i)
\]
Writing the terms in reverse order we get,

\[ S = p + (p \pm d) + (p \pm 2d) + \ldots + (a + 2d) + (a + d) + a \ldots \ldots (ii) \]

Adding (i) and (ii) we get,

\[ 2S = (a + p) + (a + p) + (a + p) + \ldots + (a + p) + (a + p) + (a + p) \]

\[ = n(a + p) \]

\[ \therefore S = \frac{n(a + p)}{2} \ldots \ldots (iii) \]

The last term \( p = n \)-th term = \( a + (n - 1) d \)

Putting the value of \( p \) in equation (iii) get,

\[ S = \frac{n}{2} (a + p) = \frac{n}{2} \{ a + a + (n \pm 1) d \} = \frac{n}{2} \{ 2a + (n \pm 1) d \} \ldots \ldots (iv) \]

Note : When the first term, the last term and the number of terms are given, sum of the series can be found by using formula (iii). On the other hand when the 1st term, the common difference and the number of terms are given, sum of the series can be found by using formula (iv).

Sometimes, \( S_n \) is written in place of \( S \) to denote the sum of \( n \) terms.

**Example 3** : Find the sum of 29 terms of the series 11 + 18 + 25 + 32 + ....

**Solution** : This is an arithmetic series whose 1st term \( a = 11 \).

Common difference \( d = 18 \pm 11 = 7 \) and the number of terms \( n = 29 \)

\[ \therefore \text{Sum, } S = \frac{n}{2} \{ 2a + (n \pm 1) d \} \]

\[ = \frac{29}{2} (2.11 + 28.7) = \frac{29}{2} (22 + 196) \]

\[ = \frac{29}{2} \times 218 = 29 \times 109 = 3161. \]

**Example 4** : 1 + 2 + 3 + .... + \( n \) = What?

**Solution** : Here 1st term \( a = 1 \), common difference \( d = 2 \pm 1 = 1 \), last term =\( n \), number of terms = \( n \).

\[ \therefore \text{Sum, } S = \frac{n}{2} (1 + n) = \frac{n(n + 1)}{2} \]

Therefore, sum of first \( n \) natural numbers = \( \frac{n(n + 1)}{2} \).
Exercise 9.1

1. Which term of 5 + 8 + 11 + ..... is 338?
2. In a certain arithmetic series, if the m-th term is m² and the n-th term is n², what is the (m + n) th term?
3. 1 + 2 + 3 + 4 + ..... + 99 = what?
4. Find the sum of n terms of the series 1 + 3 + 5 ..... 
5. 5 + 11 + 17 + 23 + .... + 59 = what?
6. 29 + 25 + 21 + ..... ÷ 23 = what?
7. 12th term of an arithmetic series is 77, what is the sum of its first 23 terms?
8. The sum of first n terms of a certain series is n(n + 1), find the series.
9. Show that, 1 + 3 + 5 + 7 + .... + 125 = 167 + 171 + 173 + .... + 209
10. If the sum of n-terms of the series 9 + 7 + 5 + ... is 144, then find the value of n.
11. Net salary of a service holder in the month of January 2000 is Tk. 10000. If his yearly increment be Tk. 300, then what will be his net salary in January 2005?
   If 10% be deducted for provident fund in every month from his net salary, then how much will be receive upto 31st January 2005?

Sum of Squares of first n Natural Numbers
It is convenient to apply some special technique to find the sum of the series 1² + 2² + 3² + ..... + n²
Suppose, S = 1² + 2² + 3² + ..... + n²
We know, r³ ÷ (r ÷ r³) = r³ ÷ (r³ ÷ 3r² + 3r ÷ 1) = 3r² ÷ 3r + 1
Here putting r = 1, 2 , 3 .......... , n we get,
1³ ÷ 0³ = 3.1² ÷ 3.1 + 1
2³ ÷ 1³ = 3.2² ÷ 3.2 + 1
3³ ÷ 2³ = 3.3² ÷ 3.3 + 1
..............................................
n³ ÷ (n ÷ 1³) = 3n² ÷ 3.n + 1
Adding, \( n^3 = 3(1^2 + 2^2 + 3^2 + \ldots + n^2) \) ð 3 \((1 + 2 + 3 + \ldots + n)\) 
\[ + (1 + 1 + \ldots + 1) \]
\[ = 3S \div \frac{3n(n+1)}{2} + n \left[ \therefore 1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2} \right] \]
\[ \therefore 3S = n^3 + \frac{3n(n+1)}{2} \] ð \( n= \frac{2n^3 + 3n^2 + 3n}{2} \)
\[ = \frac{2n^3 + 3n^2 + n}{2} = \frac{n(2n^2 + 3n + 1)}{2} \]
\[ = \frac{n(n+1)(2n+1)}{2} \]
\[ \therefore S = \frac{n(n+1)(2n+1)}{6} \]

**Sum of Cubes of first n Natural Numbers**

Suppose, \( S = 1^3 + 2^3 + 3^3 + \ldots + n^3 \)

In this case it is advantageous to apply the following technique
\[ (r + 1)^2 \cdot r^2 \div (r \div 1)^2 \]
\[ = r^2 \{(r+1)^2 \div (r \div 1)^2\} \]
\[ = r^2 \cdot 4r = 4r^3 \]

Putting \( r = 1, 2, 3, \ldots, n \), we get;
\[ 2^2 \cdot 1^2 \div 1^2 = 4.1^3 \]
\[ 3^2 \cdot 2^2 \div 2^2 = 4.2^3 \]
\[ 4^2 \cdot 3^2 \div 3^2 = 4.3^3 \]

\[ \ldots \ldots \ldots \ldots \ldots \ldots \ldots \]
\[ (n + 1)^2 \cdot n^2 \div n^2 = 4n^3 \]

Adding \( (n + 1)^2 \cdot n^2 = 4(1^3 + 2^3 + 3^3 + \ldots + n^3) = 4S \)
\[ \therefore S = \frac{n^2(n+1)^2}{4} = \left\{ \frac{n(n+1)^2}{2} \right\} \]

**N. B.**

\[ 1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2} \]
\[ 1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6} \]
\[ 1^3 + 2^3 + 3^3 + \ldots + n^3 = \frac{n^2(n+1)^2}{4} \]
\[ \therefore 1^3 + 2^3 + 3^3 + \ldots + n^3 = (1 + 2 + 3 + \ldots + n)^2 \]
**Example 5.** If sum of cubes of first \( n \) natural numbers is 225, what is the value of \( n \)? What is the sum of squares of those numbers?

**Solution:** Sum of cubes of first \( n \) natural numbers = \( \left[ \frac{n(n + 1)}{2} \right]^2 \)

According to the question \( \frac{n(n + 1)}{2} = 225 = (15)^2 \)

\[ \therefore \frac{n(n + 1)}{2} = 15 \]

or, \( n^2 + n = 30 \) or, \( n^2 + n = 30 \)

or, \( (n + 6)(n - 5) = 0 \)

\[ \therefore n = 5 \quad \text{[since } n \text{ cannot be negative]} \]

So sum of squares of those numbers

\[ = \frac{n(n + 1)(2n + 1)}{6} = \frac{5 \times 6 \times 11}{6} = 55 \]

**Geometric Series**

In the series 3 + 6 + 12 + 24 + ........

1st term = 3, 2nd term = 6, 3rd term = 12, 4th term = 24 etc.

Ratio of 1st term with the 2nd term = \( \frac{6}{3} = 2 \)

Ratio of 2nd term with the 3rd term = \( \frac{12}{6} = 2 \)

Ratio of 3rd term with the 4th term = \( \frac{24}{12} = 2 \)

In a series, if the ratio of any term with the following term is always the same, then that series is called a geometric series. The series given above is a geometric series and in this series the common ratio is 2.

Here 1st term = 3, common ratio = 2

\[ \therefore 2 \text{nd term} = 3 \times 2 = 6, \quad 3 \text{rd term} = 3 \times 2^2 = 12 \]

4th term = \( 3 \times 2^3 = 24 \)

In general the \( r \)-th term = \( 3 \times 2^{r-1} \)

Similarly, in a geometric series if the 1st term is \( a \), common ratio is \( q \), then \( r \)-th term is \( a q^{r-1} \)
Example 6. Find the common ratio and 8th term of the geometric series 4 + 12 + 36 + ......  

Solution: Here, the 1st term a = 4, the common ratio \( q = \frac{12}{4} = 3 \)  
\[ \therefore \text{the 8th term} = aq^7 = 4.3^7 = 8748 \]

Sum of n terms of a Geometric Series  
If in a geometric series, the 1st term is a and the common ratio is q, then the series upto n terms is  
\[ a + aq + aq^2 + \ldots + aq^{n-1} \]  

Suppose, \( S = a + aq + aq^2 + \ldots + aq^{n-1} \) ............(i)  
Multiplying both the sides by q we get,  
\[ Sq = aq + aq^2 + aq^3 + \ldots + aq^n \] .............(ii)  
Subtracting (ii) from (i) we get,  
\[ S - Sq = a - aq^n \]  
\[ \therefore S (1 - q) = a(q^n - 1) \]  
\[ or, S = a(q^n - 1) \] \[ 1 - q \]

N. B.: If \( q = 1 \), each term = a, and hence S = na

Example 7. Find the sum of 8th terms of the series 2 + 6 + 18 + .......  

Solution: The given series is a geometric whose 1st term = 2.  
Common ratio \( q = \frac{6}{2} = 3 \)  
Here, \( n = 8 \)  
\[ \therefore \text{Sum, } S = \frac{a(q^n - 1)}{q - 1} = \frac{2(3^8 - 1)}{3 - 1} = 3^8 - 1 = 6560 \]
Example 8. If in a geometric series 1st and 2nd terms are respectively 125 and 25, find the 5th and the 6th term.

Solution : Here 1st term \( a = 125 \), 2nd term = 25

\[ \therefore \text{Common ratio } q = \frac{25}{125} = \frac{1}{5} \]

\[ \therefore 5\text{th term } = aq^4 = 125 \left( \frac{1}{5} \right)^4 = 125 \cdot \frac{1}{5^3 \cdot 5} = \frac{1}{5} \]

6th term = \( aq^5 = 125 \cdot \left( \frac{1}{5} \right)^5 = \frac{1}{5^2} = \frac{1}{25} \)

Example 9. Find the sum of first 10 terms of the series \( 3 \cdot 6 + 12 \) ...........

Solution : The given series is a geometric series where 1st term \( a = 3 \).

common ratio \( q = \frac{6}{3} = \frac{2}{1} > 1 \)

Here the number of terms \( n = 10 \).

\[ \therefore \text{Sum of first 10 terms } = \frac{a(1 - q^n)}{1 - q} = \frac{3 \left( 1 - \left( \frac{2}{1} \right)^{10} \right)}{1 - \left( \frac{2}{1} \right)} \]

\[ = \frac{3(1 - 1024)}{3} = 1023 \]

Example 10. Find the sum of first 5 terms of the series

\[ 1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots \]

Solution : The given series is a geometric series where 1st term \( a = 1 \).

Common ratio \( q = \frac{1}{3} = \frac{1}{3} < 1 \), Number of terms \( n = 5 \).

\[ \therefore \text{Sum of first five terms } = \frac{a(1 - q^n)}{1 - q} = \frac{1 \left( 1 - \left( \frac{1}{3} \right)^5 \right)}{1 - \frac{1}{3}} = \frac{1 - \frac{1}{243}}{\frac{2}{3}} \]

\[ = \frac{3}{2} \times \frac{(243 \cdot 1)}{243} = \frac{3}{2} \times \frac{242}{243} = \frac{121}{81} \].
Exercise 9.2

1. Show that, \(1^3 + 2^3 + 3^3 + 4^3 + \ldots + 10^3 = (1 + 2 + 3 + 4 + \ldots + 10)^2\)

2. If \(\frac{1^3 + 2^3 + 3^3 + 4^3 + \ldots + n^3}{1 + 2 + 3 + 4 + \ldots + n} = 210\), what is the value of \(n\)?

3. What is the 9th term of the series 128 + 64 + 32 + .........?

4. What term of the series \(\frac{1}{\sqrt{2}} \div 1 + \sqrt{2} \div \ldots \ldots \ldots \text{is } 8\sqrt{2}\) ?

5. If the 5th term of geometric series is \(\frac{2\sqrt{3}}{9}\) and the 10th term is \(\frac{8\sqrt{2}}{81}\), find the 3rd term of the series.

6. If 5 + x + y + 135 is a geometric series, find the value of \(x\) and \(y\).

7. Find the sum of first 8 terms of the series 1 + \(\frac{1}{2}\) + \(\frac{1}{4}\) + \(\frac{1}{8}\) + .........

8. What is the sum of first 7 terms of the series 2 ÷ 4 + 8 ÷ 16 + .........?

9. Find the sum of (2\(n\) + 1) terms of the series 1 ÷ 1 + 1 ÷ 1 + 1 ÷ ..... 

10. What is the sum of first 10 terms of the series \(\log 2 + \log 4 + \log 8 + \ldots \) ?

11. What is the sum of the series 6 + 12 + 24 + .........+ 384?

12. If the sum of the \(n\)-terms of the series 2 + 4 + 8 + 16 + .... is 254, then what is the value of \(n\)?

13. An iron rod of length 1 metre was divided into 10 pieces, so that the length of the pieces formed geometric series. If the length of biggest piece is 10 times the smallest piece, then find the length of smallest piece in approximate millimetre.
Multiple Choice Questions (MCQ) :

1. Which of the following is the formula for determining the sum of first $n$ natural numbers?
   A. $\frac{n(n+1)}{2}$
   B. $\frac{n(n + 1) (2n + 1)}{6}$
   C. $\frac{n(n+1)}{2}$
   D. $\left[ \frac{n(n+1)}{2} \right]^2$

2. Which of the following relations is true if the series $x + y + z + w + \ldots$ is a geometric series?
   A. $\frac{y}{x} = \frac{w}{z}$
   B. $y \div x = w \div z$
   C. $\frac{x}{y} = \frac{w}{z}$
   D. $x \div y = z \div w$

3. Which of the following is 21st term of the series $a \div a + a \div a + \ldots$?
   A. $-a$
   B. $a$
   C. $21a$
   D. $-21a$

4. Look at the following sentences :
   i. $\frac{n(n+1)(2n+1)}{6}$ is the sum of the square of the 1st $n$ natural numbers.
   ii. If $r > 1$, then $a + ar + ar^2 + \ldots \ldots + ar^{n-1} = \frac{a(r^n - 1)}{r-1}$
   iii. $n^{th}$ term of a geometric series is $ar^n$.
   As per above sentences which of the following is correct?
   A. i and iii
   B. i and ii
   C. ii and iii
   D. i, ii and iii

Answer questions no. (5 Ñ 7) based on the following information :

5. Which of the following is the common difference of the series?
   A. $\log 3$
   B. $\log 9$
   C. $2 \log 3$
   D. $3 \log 3$

6. What is the 10th term of the series?
   A. $\log 1000$
   B. $\log 9000$
   C. $\log 72900$
   D. $\log 59049$
7. What is the sum of the first 15 numbers of the series?
   A. 12 log 3  
   B. 15 log 3  
   C. 120 log 3  
   D. 150 log 3

**Creative Questions :**

1. First term of a geometric series is a, common ratio is r, fifth term is $3\sqrt{3}$ and eighth term is $\text{D}$ 27.
   A. State the above information in 2 equations.  
   B. Determine the 15th term of the series.  
   C. Find the series and determine the sum of the first 11 terms.

2. A government officer obtain Tk. 10,000 as his basic salary in January, 2001. Yearly increment of his monthly salary is Tk. 400.
   A. State his monthly salary in arithmetic series.  
   B. Determine the monthly basic salary of January, 2006 by solving the arithmetic series.  
   C. If provident fund is deducted at the rate of 15% from his basic salary then determine the total deduction for provident fund in 25 years.
Answer

Exercise 1.1

1. (i) ∈ (ii) ∉ (iii) ∈ (iv) ∉ (v) ∉
2. (i) ⊆ (ii) ⊈ (iii) ⊆ (iv) ⊆
3. (i) {4} (ii) {1, 2, 3, 0, 1, 2, 3} (iii) ∅ (iv) {2, 4, 6, 8} (v) {1, 2, 3, 6, 7, 14, 21, 42} (vi) {3, 6, 9, 12, 15, 18}
4. (i) A = \{1, 3, 5, 7, 9, 15, 21, 35, 45, 63, 105, 315\} B = \{1, 3, 5, 7, 15, 21, 25, 35, 75, 105, 175, 525\}
   (ii) {36} (iii) ∅
5. A ∪ B = \{1, 2, 3, a, b\} , A ∩ B = \{3\}
6. Best answers : \{1, 0, 1\}, \{1, 0, 2\}, \{0, 1, 2\} 7. ∅
8. A ∪ B = \{1, 2, 3\} A ∩ B = ∅
9. (i) \{1, 3, 5\} (ii) \{3, 5\} (iii) \{2, 4, 6\}
   (iv) \{1, 3, 5\} (v) \{1, 2, 4, 6\} (iv) \{\}\n12. 1% 13. 4, 13

Exercise 1.2

1. \(P(B) = \{\{1, 2\}, \{1\}, \{2\}, \emptyset\}\)
2. \(P(C) = \{\{a, b, c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a\}, \{b\}, \{c\}, \emptyset\}\)
3. \(x = 2, y = 1\)  4. (2, 3)
5. \(A \times B = \{(0, 1), (0, 2), (1, 1), (1, 2)\}\)
   \(B \times A = \{(1, 0), (1, 1), (2, 0), (2, 1)\}\)
6. \(A \times B = \{(a, p), (a, q), (b, p), (b, q), (c, p), (c, q)\}\)
   \(B \times A = \{(p, a), (p, b), (p, c), (q, a), (q, b), (q, c)\}\)
7. \(A \times (B \cup C) = \{(a, 2), (a, 3), (a, 4), (b, 2), (b, 3), (b, 4)\}\)
   \(A \times (B \cap C) = \{(a, 3), (b, 3)\}\)
8. \(A \times B = \{(a, 9)\}, B \times A = \{(0, a)\},\)
9. \(\{(1, \frac{1}{2}), (1, \frac{1}{3}), (\frac{1}{2}, 1), (\frac{1}{3}, 1)\}\)
10. \(A \times B = \{(c, \ell), (c, g), (\ell, \ell), (\ell, g), (f, \ell), (f, g)\}\)
11. \{Anu, Rahi\}, \{Anu, Masha\}, \{Shumon, Rahi\}, \{Shumon, Masha\}, \{Mim, Rahi\}, \{Mim, Masha\}\}
12. \{(Akram, Bulbul), (Akram, Nannu), (Bulbul, Nannu), (Bulbul, Akram), (Nannu, Akram), (Nannu, Bulbul)\}
Exercise 2

1. Do yourself (number line)
   (i) 4 ≤ x ≤ 12  (ii) 4 ≤ x < 24  (iii) 0 ≤ x < 41

2. (i) \{x \in \mathbb{R} : 4 \leq x \leq 4\}
   (ii) \{x \in \mathbb{R} : 1 < x < 2 \text{ or } 2 < x \leq 4\}
   (iii) \{\sqrt{2}, \sqrt{2}\}
   (iv) \{10, \sqrt{10}\}

3. (i) 1  (ii) 7  (iii) 10

4. (i) \{x \in \mathbb{R} : 1 < x < 9\}  (ii) \{1, 9\}  (iii) \{x \in \mathbb{R} : x < 1 \text{ or } x > 9\}

5. Do yourself [Many answers are possible]

6. Do yourself [Many answers are possible]

7. Do yourself [Many answers are possible]

8. (i) \{x : 3 < x < \frac{5}{3}\}  (ii) \left[\frac{13}{2}, \frac{17}{4}\right]

9. 0 ≤ 318 ≤ 10  2 ≤ 4392

11. (i) 5 ≤ 5451  (ii) 0 ≤ 1010

Exercise 3.1

1. (i) a^2 + 6ab + 9b^2  (ii) a^2b^2 + 2abc + c^2  (iii) x^4 + \frac{4x^2}{y^2} + \frac{4}{y^4}
   (iv) 9p^2 + 16q^2 + 25r^2 + 24pq ≤ 40qr ≤ 30pr
   (v) \frac{a^2}{4} + \frac{4}{b^2} + \frac{1}{c^2} + \frac{2a}{b} + \frac{4}{bc} + \frac{a}{c}
   (vi) 992016  (vii) a^2x^2 + b^2y^2 + c^2z^2 + 2abxy + 2bcyz ≤ 2caxz

2. (i) 196y^2  (ii) (2a + 2b)^2  (iii) 2u ≤ 25

3. 50

4. a^2 + 2

5. ± p

6. ± 4

7. 2

8. 1

9. (i) 74  (ii) 35

11. Many answers are possible, for example, 23^2 ≤ 22^2, 9^2 ≤ 6^2, 7^2 ≤ 2^2 etc.

12. 71

13. 2p^2 ≤ 2q

14. 14

15. c

19. 10

20. 0

21. (x + 4)^2 ≤ 6^2
Exercise 3.2

1. (i) \( abc + (ab + bc + ca) \) \( x + (a + b + c) \) \( x^2 + x^3 \)
   (ii) \( 24 + 26x + 9x^2 + x^3 \)
2. (i) \( 27x^3 \mp 108x^2y + 144xy^2 \mp 64y^3 \)
   (ii) \( a^3 \mp b^3 + c^3 \mp 3a^2b + 3a^2c + 3ab^2 + 3ac^2 + 3b^2c \mp 3bc^2 \mp 6abc \)
   (iii) 65450827
3. (i) \( 2(x^3 + y^3 + z^3) \)
   (ii) \( 8a^3 \)
   (iii) \( 8(b + c)^3 \)
4. 8 5.9 6.54 8.0 9.665 11.39 12.\( \frac{79}{3} \), 135
13. 34 14. \( 18\sqrt{3} \)

Exercise 3.3

1. \( 3ab(a + 2b + 4ab) \)
2. \((x + 5y)(a + 3b) \)
3. \((a + b)(x + y) \)
4. \((1 + a)(1 + b) \)
5. \((a \mp 1)(b + 1) \)
6. \((a \mp b + c)(a \mp b \mp c) \)
7. \((ax + by + ay \mp bx)(ax + by \mp ay + bx) \)
8. \((a + b \mp 3c)(a + b \mp 3c + 1)(a + b \mp 3c \mp 1) \)
9. \((2x + y \mp z)(2x \mp y + z) \)
10. \((a^2 + 2a + 2)(a^2 \mp 2a + 2) \)
11. \((x^2 + 3x + 5)(x^2 \mp 3x + 5) \)
12. \(3(2a^2 + 2ab + b^2)(2a^2 \mp 2ab + b^2) \)
13. \((a \mp b)(a + b \mp 2c) \)
14. \((x^2 + 2x + 3)(x^2 \mp 2x + 3) \)
15. \((a^2 + 5a \mp 1)(a^2 \mp 5a \mp 1) \)
16. \((c + a \mp b)(c \mp a + b) \)
17. \((a + b \mp 1)(a \mp b + 1) \)
18. \((R \mp r)(R \mp 3r) \)
19. \((a + 2)(a^2 \mp 2a + 4) \)
20. \(m(m \mp 2)(m^2 + 2m + 4) \)
21. \((x + 2)(x^2 + x + 1) \)
22. \((2a \mp a + b)(4 + a^2 + b^2 + 2a \mp 2b + 2ab) \)
23. \((a \mp b)(2a^2 + 5ab + 8b^2) \)
24. \(mn(m \mp n) \)
25. \((y + 1)(a \mp y \mp 1) \)
26. \(\sqrt{2x}(1 + \sqrt{2x}) \)
27. \((x + \sqrt{3})(x^2 \mp \sqrt{3}x + 3) \)
28. \(A(R \mp r)(R^2 + Rr + r^2 + hR + hr) \)
29. \((x + a + 2)(x \mp a + 1) \)
30. \((x^2 + 3x \mp 2) \)
31. \((4x \mp 5y)(4x + 5y \mp 2) \)
32. \(4\pi r(3R^2 + 3Rr + r^2) \)
33. \(\frac{1}{2}mu(2v + 3u) \)
34. \((\sqrt{2x + 5})(2x^2 \mp 5\sqrt{2x} + 25) \)
Exercise 3.4

1. $(x + 5) (x - 4)$
2. $(x - 10) (x + 2)$
3. $(x - 10) (x + 2)$
4. $(x - 20) (x + 1)$
5. $(x - 18) (x - 12)$
6. $(u + 5) (u + 1)$
7. $(a^2 + 5) (a + 1)$
8. $(x^2 - 8) (x^2 - 2)$
9. $(x^3 - 4) (x^3 - 3)$
10. $(u - a + b) (u - a - b)$
11. $(x + y - 6) (x + y + 2)$
12. $(x + 3a) (x - 2a)$
13. $(x - a)$
14. $(x^2 + 2x + 15) (x + 3)$
15. $(x + a + 1) (x - a - 1)$
16. $x (x + 3) (x^2 - 1)$

Exercise 3.5

1. $(4a + 3) (a + 2)$
2. $(7p - 8) (p + 1)$
3. $(7x + 4) (5x - 3)$
4. $(5x - 3y) (3x + 7y)$
5. $(x - 1) (ax + bx - a + b)$
6. $(x + ay + y) (ax - x + y)$
7. $(7x - 2) (x + 3)$
8. $(p - 6) (6p + 25)$
9. $2(6x^2 + 10x + 1) (12x^2 + 20x + 9)$
10. $(x + y) (ax - mx + my - xy)$
11. $(x - 2) (x^2 + x + 2)$
12. $(x + 3) (x^2 + x + 1)$
13. $(x + 2) (4x - 3)$
14. $(a^2 + 3a + 5) (a^2 + 3a - 3)$
15. $(x^2 - 3x - 6) (x^2 - 3x - 16)$

Exercise 3.6

1. $(a + 1) (a - 6) (a - 4)$
2. $(x + 1) (x + 2) (x + 3)$
3. $(a - b) (a^2 - 2ab + b^2)$
4. $(x + 3) (x^2 - 3x + 12)$
5. $(a - 1) (a^2 + 2a + 3)$
6. $(2a - 1) (a^2 + a + 1)$
7. $(x - 2) (x^2 + x + 2)$
8. $(x(x - 1) (x^2 + x + 1)$
9. $(x + 3y) (x + y) (x + 2y)$
10. $(3 + x) (2 + x) (2 - x)$
11. $(x - 2) (2x + 1) (x^2 + 1)$
12. $(a + 1) (3a^2 - 3a + 5)$
Exercise 3.7

1. \( x + 1 \)  
2. \( 1 \)  
3. \( a + b + c \)  
4. \( x \cdot 5 \)  
5. \( (x + 2) (x + 1) (x \cdot 1) \)  
6. \( x^6 \cdot 1 \)  
7. \( a x (x^2 \cdot x^2) (x^2 \cdot c^2) \)  
8. \( (x \cdot 3) (x^2 + 2x + 3) (x^2 + x + 1) \)  
9. \( 36(x^2 \cdot 1) (x^2 \cdot 4) (x^2 \cdot 9) \)  
10. \( x^2 (x \cdot 2) (x + 2) (x + 4) \)

Exercise 3.8

1. Tk. 49  
2. Tk. \( C \left( 1 + \frac{r}{100} \right) \)  
3. Tk. \( \frac{100p}{100 + x} \)  
4. Tk. 25  
5. 7649089  
6. Tk. 3ú81  
7. Tk. 625  
8. Tk. 625  
9. Tk. \( \frac{1400}{100 + x} \)  
10. Tk. 450  
11. \( \frac{n(100 \cdot D \cdot r)}{100 + s} \)  
12. \( \frac{12(100 \cdot D \cdot x)}{100 + 11x} \)  
13. 60 hours  
14. \( \frac{pq(r + s)}{r(p + q)} \) minutes.  
15. \( \frac{2}{3} (p + r) \) days  
16. \( \frac{mn}{n \cdot D \cdot m} \) days  
17. Tk. \( x \left( 1 + \frac{y}{100} \right) \)  
18. \( \frac{100y}{100 + y} \)  
19. The speed of Ashique is \( \frac{d}{t_1 + t_2} \) km/hour, the speed of Razib is \( \frac{dt_2}{(t_1 + t_2)t_1} \) km/hour.  
20. Tk. \( \left\{ \frac{px}{100 + x} \right\} \), Tk. 300  
21. Bill Tk. 510ú72, Vat Tk. 66ú62  
22. 50 persons, Tk. 48  
23. The speed of boat is \( \frac{d}{2} \left( \frac{t_1}{t_2} + \frac{1}{t_1} \right) \) km/hour, the speed of current is \( \frac{d}{2} \left( \frac{1}{t_2} \cdot D \cdot \frac{1}{t_1} \right) \) km/hour.

Exercise 4.1

1. \( \frac{ab}{a + b} \)  
2. 1  
3. 1  
4. 1  
5. (i) \( \pi^2 \)  
   (ii) 1  
   (iii) \( 2^n + 1 \)  
6. 4  
7. 4  
8. \( \frac{1}{50} \)  
9. \( \frac{1}{9} \)
**Exercise 4.2**

1. (i) 4  (ii) $\frac{3}{2}$  (iii) 4  (iv) $\frac{1}{2}$  (v) $\frac{1}{2}$  (vi) $\frac{1}{3}$  (vii) $\frac{5}{6}$

2. (i) 100  (ii) 0.01  (iii) 125  (iv) 25  (v) 5  (vi) 3

**Exercise 4.3**

6. (i) $\log_{2}$  (ii) $2 \log_{5}$  (iii) $\log_{2}$  (iv) $\frac{3}{2}$  (v) 0

**Exercise 4.4**

1. $7 \times 10^2$  2. $1 \times 10^{10}$  3. $8 \times 10^2$  4. $2 \times 10^{10}$

5. $5 \times 10^6$  6. $6 \times 10^{11}$  7. $1 \times 10^8$  8. $4 \times 10^9$

9. 1000  10. 0.000001  11. 12300  12. 0.009873

13. 0.000000132  14. 0.0000003356

**Exercise 4.5**

1. (i) 2  (ii) 1  (iii) 0  (iv) 0  (v) $\frac{p}{s}$  (vi) $\frac{p}{s}$  2. (i) 2.51054  (ii) 0.96708

3. (i) 2.63468  3. (i) 3.0697  (ii) 346.74  (iii) 0.039902  4. (i) 36.7921

5. (i) 83.366  (ii) 40.1458  5. (i) 10.6558  (ii) 10.3817

6. Tk. 481.13 (approx)  7. 14.2 years (approx)  8. 200 metres

9. (i) $\frac{p}{s}$  10. (i) 0.07781  (ii) 1.3221  (iii) 1.6231

**Exercise 5.1**

1. $a^2 \cdot b^2$  2. $\sqrt{\pi} \cdot \Omega$  or $\sqrt{22} \cdot \Omega \sqrt{7}$  3. 45, 60  4. 1 $\Omega$

5. $1 \cdot \Omega \cdot 4$  6. 20%  7. 18 $\Omega \cdot 25$  8. Father 35 years, son 10 years

9. $(t_1 + t_2) \cdot t_1$  10. $\left(\frac{p}{s} + 1\right)$ metre
12. (i) $\frac{3}{4}$  
(ii) $\frac{2ab}{b^2 + 1}$  
(iii) $0, \pm \frac{1}{a} \sqrt{\frac{2a}{b}}$  
(iv) $b$

23. (i) $\frac{4a}{a^2 + 4}$  
29. (i) 10  
(ii) $\frac{b}{2a} \left( c + \frac{1}{c} \right)$  
(iii) $\frac{1}{2}$, 2.

Exercise 5.2

1. Aziz Tk. 300, Abed Tk. 240, Ashique Tk. 320
2. A Tk. 40, B. Tk. 60, C. Tk. 120. D Tk. 80
3. 200, 240, 250
4. Bulbul 81 runs, Nannu 54 runs, Akram 36 runs.
5. Officer Tk. 8000, Clerk Tk. 4000, Peon Tk. 2000.
6. Tk. 7200 7. 70 8. 20% 9. 50% 10. 21%
11. 24% 12. 70% 13. 53½2 quintal 14. 4Ω9 15. 70%
16. 1176 sq. metre 17. 13 Öl2 18. 4ü5 cm., 6 cm., 7ú5 cm.
19. Tk. 210, Tk. 224 and Tk. 240. 20. 120.

Exercise 6.1

1. 4 2. ab 3. $\frac{5}{2}$ 4. $\frac{7}{2}$ 5. 2 $(1 + \sqrt{3})$ 6. $\sqrt{5}$ 7. 6
8. $a + b$ 9. $\frac{a + b}{2}$ 10. $\frac{3}{5}$ 11. $\{\sqrt{a}\}$ 12. $\left[\frac{a + b}{2}\right]$ 13. $\{\sqrt{a^2 + b^2 + c^2}\}$
14. $\phi$ 15. $\{2\}$ 16. $\{3\}$ 17. $\left[\frac{p + q}{2}\right]$ 18. $\left[\frac{\sqrt{1}}{3}\right]$ 19. $\phi$ 20. $\phi$

Exercise 6.2

1. 60, 40 2. 5 3. $\frac{3}{4}$ 4. 9 5. 50¡ 7. 72
8. The number of coins of twenty five paisa is 100, the number of coins ten paisa is 20
9. 120 km 10. 60 11. 100 12. Tk. 3200
Exercise 6.3

1. \( y < 8 \)

2. \( x < 4 \)

3. \( x > 1 \)

4. \( z \leq 6 \)

5. \( x \geq 3 \)

6. \( x \leq 6 \)

7. \( t \geq 3 \)

8. \( x > 1 \)

Exercise 6.4

1. \( 3x + \frac{x + 2}{2} < 29, \ 0 < x < 8 \)

2. \( 4x + x \leq 40, \ 0 < x \leq \frac{43}{5} \)

3. \( 30x + 20x < 500, \ 0 < x < 10 \)

4. \( \frac{x + x + 120}{9} \leq 100; \ 0 < x \leq 390 \)

5. \( 5x < 40, \ 5 < x < 8 \)

6. Father's age is \( \leq 42 \) years

7. If the present age of Nadira is \( x \) years, then \( 14 < x < 17 \)

8. If the time is \( t \) seconds, then \( t \geq 50 \)

9. If the time of flight is \( t \) hours, then \( t \geq 6\frac{1}{4} \)

10. If the time of flight is \( t \) hours, then \( t \geq 5 \)

11. If the number is \( x \), then \( 0 < x < 5 \).

Exercise 6.5

1. \( \{\mathbb{D} \ 1, \ \mathbb{D} \ 2\} \)

2. \( \{\mathbb{D} \ 3, \sqrt{5}\} \)

3. \( \left\{ \frac{3\sqrt{2}}{2} \hat{a} \frac{\sqrt{10}}{2} \right\} \)

4. \( \{\mathbb{D} \ 6 \hat{a} \frac{3}{2}\} \)

5. \( \{1, 10\} \)

6. \( \left\{ \frac{3}{4} \hat{a} \frac{4}{3} \right\} \)

7. \( \{\mathbb{D} \frac{3}{20} \hat{a} 2\} \)

8. \( \{\mathbb{D} \frac{2}{3} \hat{a} 2\} \)

9. \( \left\{ 3 \hat{a} \mathbb{D} \frac{1}{2} \right\} \)

10. \( \{0, a + b\} \)

11. \( \left\{ \frac{1}{2} \hat{a} 2 \right\} \)

12. \( \{7, \mathbb{D} \ 7\} \)
13. \( \{ \sqrt{ab} \} \) 14. \( \{1, \ddot{D} 1\} \) 15. \( \{ \ddot{D} a, \ddot{D} b\} \) 16. \( \{3a, 2a\} \)
17. \( \left[ \frac{1}{3} \right] \) 18. \( \{1\} \) 19. \( \{1, 4\} \) 20. \( \{0, 4a\} \)

**Exercise 6.6**

1. 56 metres 2. 9 3. \( \frac{11}{13} \) 4. 16 metres 5. 27 metres 6. 5 metres
7. 20 8. 84 or, 48 9. 15
10. Length 21 metres, breadth 11 metres 11. 30 sq. cm 12. 17
13. 16 cm 14. 17 or, 70 15. 70

**Exercise 6.7**

1. \( \{x \in \mathbb{R} : x > 3 \text{ or } x < 2\} \) 2. \( \{x \in \mathbb{R} : x \geq 1 \text{ or } x \leq \ddot{D} 2\} \)
3. \( \{x \in \mathbb{R} : x > \frac{1}{2} \text{ or } x < \ddot{D} 2\} \) 4. \( \{x \in \mathbb{R} : x \neq 1\} \)
5. \( \{x \in \mathbb{R} : x > 7 \text{ or } x < \ddot{D} 1\} \) 6. \( \{x \in \mathbb{R} : x > 5 \text{ or } x < \ddot{D} 3\} \)
7. \( \{x \in \mathbb{R} : x < 3 \text{ or } x > 5\} \) 8. \( \{x \in \mathbb{R} : 1 \leq x \leq 8\} \)
9. \( \{x \in \mathbb{R} : \frac{6}{5} < x < 3\} \) 10. \( \{x \in \mathbb{R} : \frac{1}{2} < x < 1\} \)
Exercise 6.8
1. 1, 10  
2. 18, 20  
3. 25, 26  
4. 2, 7  
5. \{1, 4, 5, 6, 7, 8, 9\}

Exercise 7.1
1. \{(5, 4), (6, 4), (6, 5)\}  
2. \{(3, 5), (4, 5)\}

Exercise 7.2
1. 10, \frac{145}{27}  
2. 2 or, 3  
3. 2  
4. 22  
5. 3x

Exercise 7.3
1. Do yourself  
2. Do yourself  
3. 13 units  
4. \(x^2 + y^2 + 8x + 6y = 0\)  
5-8. Do yourself  
9. Draw the graph ; \(\sqrt{41}\) units.

Exercise 7.4
1. 14  
2. \(27x^2 \equiv 4y^3 = 0\)  
6. \(22t^2 \equiv 15rt + 2 = 0\)  
7. 6√2 metres  
8. 53û9 metres.

Exercise 8.1
1. (i) Inconsistent, no solution  
(ii) Consistent; infinitely many solutions  
(iii) Consistent; unique solution.

2. (i) Infinitely many solutions  
(ii) Unique solution  
(iii) No solution  
(iv) Unique solution  
(v) Unique solution

Exercise 8.2
1. (3, 2)  
2. (4, 1)  
3. (1, 2)  
4. (2, 6)  
5. \(\left(\frac{6}{5}, \frac{6}{5}\right)\)  
6. (2, 3)  
7. (16, 4)

8. \((a + b, b \equiv a)\)  
9. \((a + b, b \equiv a)\)  
10. \((a, b)\)  
11. (1, 1)  
12. (12, 4)
Exercise 8.3
1. (2, 1)  2. (1, 5)  3. (4, 1)  4. (12, 6)  5. \( \left( \frac{1}{4} \right) \)  6. (6, 2)
7. \( \left( \frac{1}{4}, 6 \right) \)  8. (2, 1)  9. (2, 3)  10. \( \left( \frac{ab}{a + b}, \frac{ab}{a + b} \right) \)
11. \( \left( \frac{ab}{a + b}, \frac{ab}{a + b} \right) \)
12. \( \left( \frac{c(b \cdot c)}{a(b \cdot a)^2}, \frac{c(c \cdot a)}{b(b \cdot a)} \right) \)

Exercise 8.4
1. \( \left( D \frac{1}{2}, 4 \right) \)  2. (3, 2)  3. (2, 3)  4. (1, 2)  5. (c, a)
6. (a, b)  7. (a, b)  8. (5, 4)  9. (2, 4)  10. (4, 5)
[in each case verify the solution]

Exercise 8.5
1. (2, 3)  2. (D 7, 3)  3. (4, 5)  4. (a + b, b \cdot a)  5. (1, D 1)  6. (a, b)  7. (2, 3)
8. \( (a^2, b^2) \)  9. \( \left( \frac{a}{a^2 + b^2}, \frac{b}{a^2 + b^2} \right) \)
10. (0, 2b)  11. (a, b)
12. \( \left( \frac{b \cdot c}{a \cdot b}, \frac{c \cdot D}{a \cdot D} \right) \)
13. (a, b)  14. \( (a^2, b^2) \)

Exercise 8.6
1. (2, 1)  2. (1, 1)  3. \( \left( 1, \frac{1}{2} \right) \)  4. (3ú5, 2ú5)
5. No solution  6. No solution  7. (2ú2, 1ú4)

Exercise 8.7
1. \( \frac{15}{26} \)  2. \( \frac{5}{7} \)  3. 51  4. 54  5. 73 or, 37  6. 27
7. The age of father is 33 years, the age of the son is 11 years.
8. 50 years  9. y = 42; 10 years  10. x = 3, y = 4
11. 5 km/hour  12. 5  13. Length 17 metres, breadth 9 metres
14. Length 25 metres, breadth 12 metres  15. x = 10, y = 70
16. x = 20, y = 40  18. Tk. 3000, Tk. 125  19. 30 ml, 70 ml.
Exercise 8.8

[Solutions are to be considered as \((x, y)\)]

1. \((4, 3), (0, 5)\)  
2. \((1, 1), \left(\frac{1}{3}, \frac{5}{3}\right)\)  
3. \((5, 6), (6, 5), (5, 6), (6, 5)\)

4. \((7, 6), (6, 7), (6, 7), (6, 7)\)  
5. \(\left(\frac{1}{2}, 6\right), (3, 1)\)  
6. \((7, 2)\)

7. \((10, 1), (2, 8), (8, 2)\)  
8. \((3, 1), \left(\frac{2}{3}, \frac{25}{3}\right)\)  
9. \((4, 1), (1, 4)\)

Exercise 8.9

1. 16 metres, 15 metres  
2. 13, 9  
3. 5  
4. 19  
5. Length 6 metres, breadth 4 metres, or length 16 metres, breadth \(\frac{1}{2}\) metres.

6. Length 25 metres, breadth 24 metres  
7. Length 8 metres, breadth 6 metres  
8. 36  
9. \(8\sqrt{3}\) metres  
10. Length 20 metres, breadth 15 metres.

Exercise 9.1

1. 127  
2. \(m^2 + mn + n^2\)  
3. 4950  
4. \(n^2\)  
5. 320  
6. 42  
7. 1771  
8. \(2 + 4 + 6 + \ldots\)  
9. \(1\)  
10. 18  
11. Tk. 11, 500, Tk. 5, 83, 940

Exercise 9.2

2. 20  
3. \(\frac{1}{2}\)  
4. 9th term  
5. \(\frac{1}{\sqrt{3}}\)  
6. \(x = 15, y = 45\)  
7. \(\frac{127}{128}\)  
8. 86  
9. 1  
10. 55 \(\log_2\)  
11. 762  
12. 7  
13. 21 millilitre.
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